Warsaw University of Technology

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Institute of Aeronautics and Applied Mechanics

Finite element method (FEM1)

Part 1. Introduction

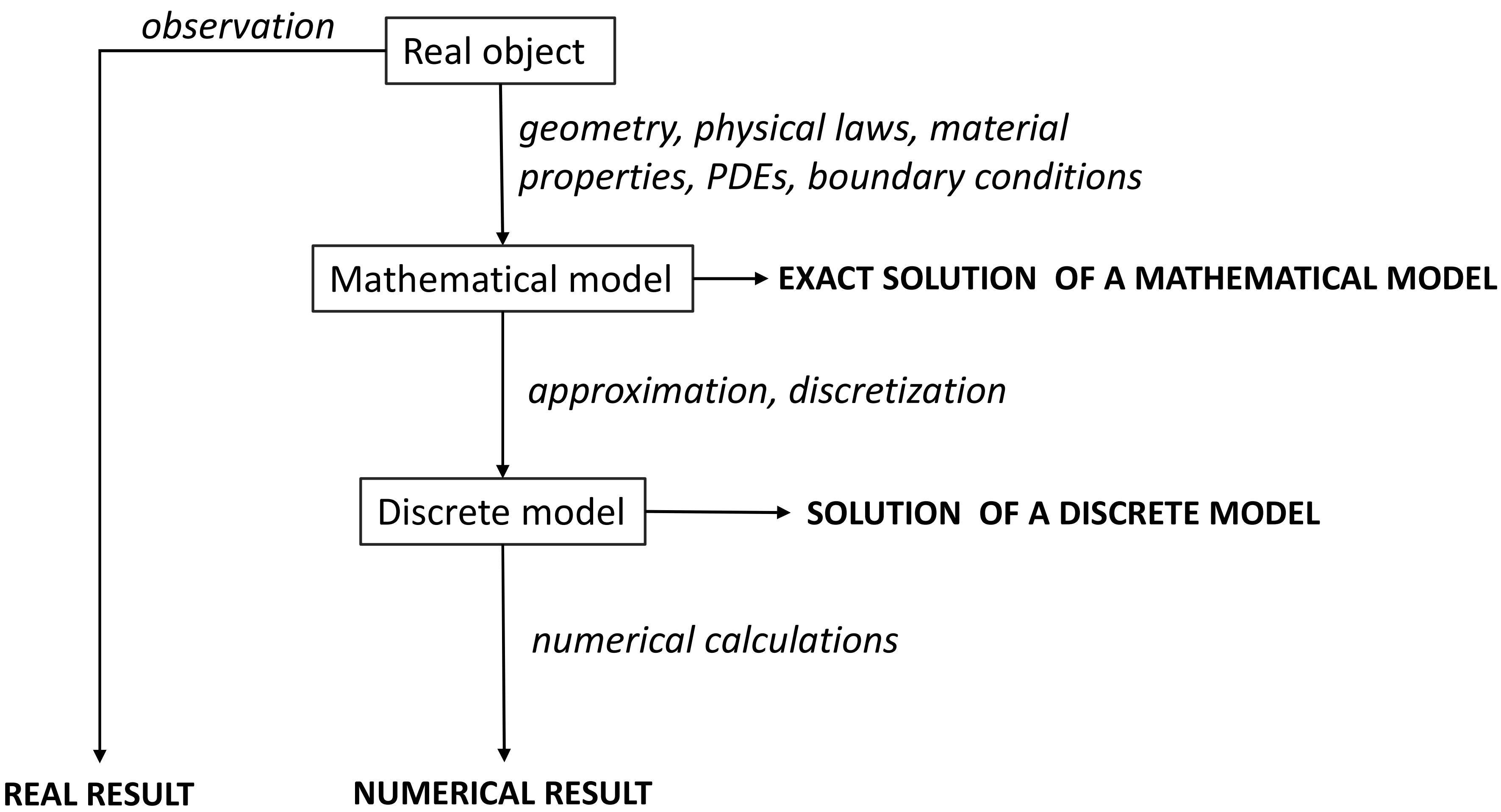
02.2020

The finite element method (FEM) is an approximate method which can be used as a numerical procedure to solve physical problems including:

- solid body mechanics,
- heat transfer,
- fluid flow,
- electromagnetism,
- coupled field problems
- …

FEM was developed in 1950s to solve problems for the civil and aeronautical industries. The method become the most powerful analysis tool, mainly due to the development of computers.

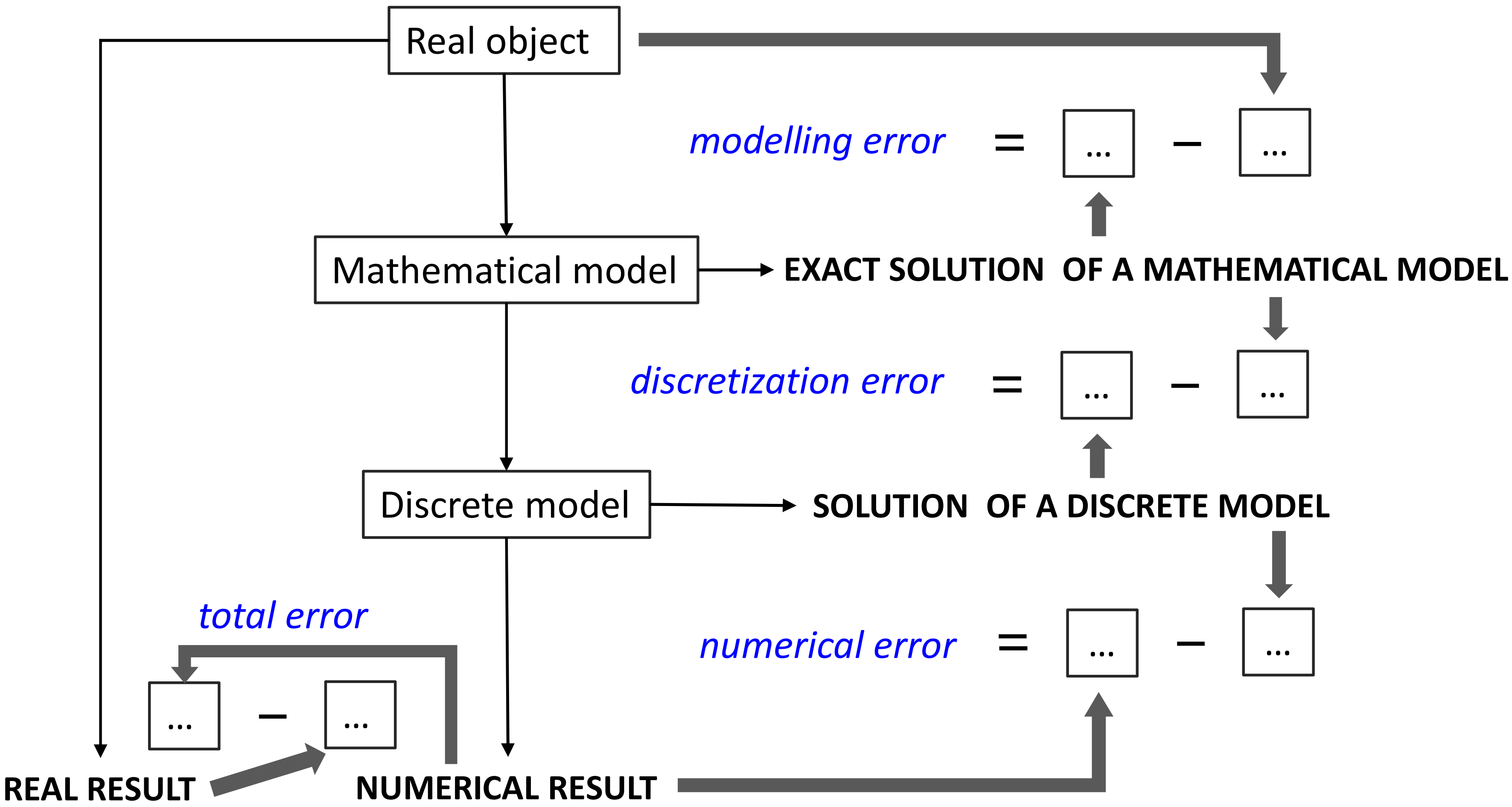
The aim of the lecture is to supply basic knowledge and skills required for understanding and application of the FEM to solve boundary value problems for partial differential equations (PDEs).



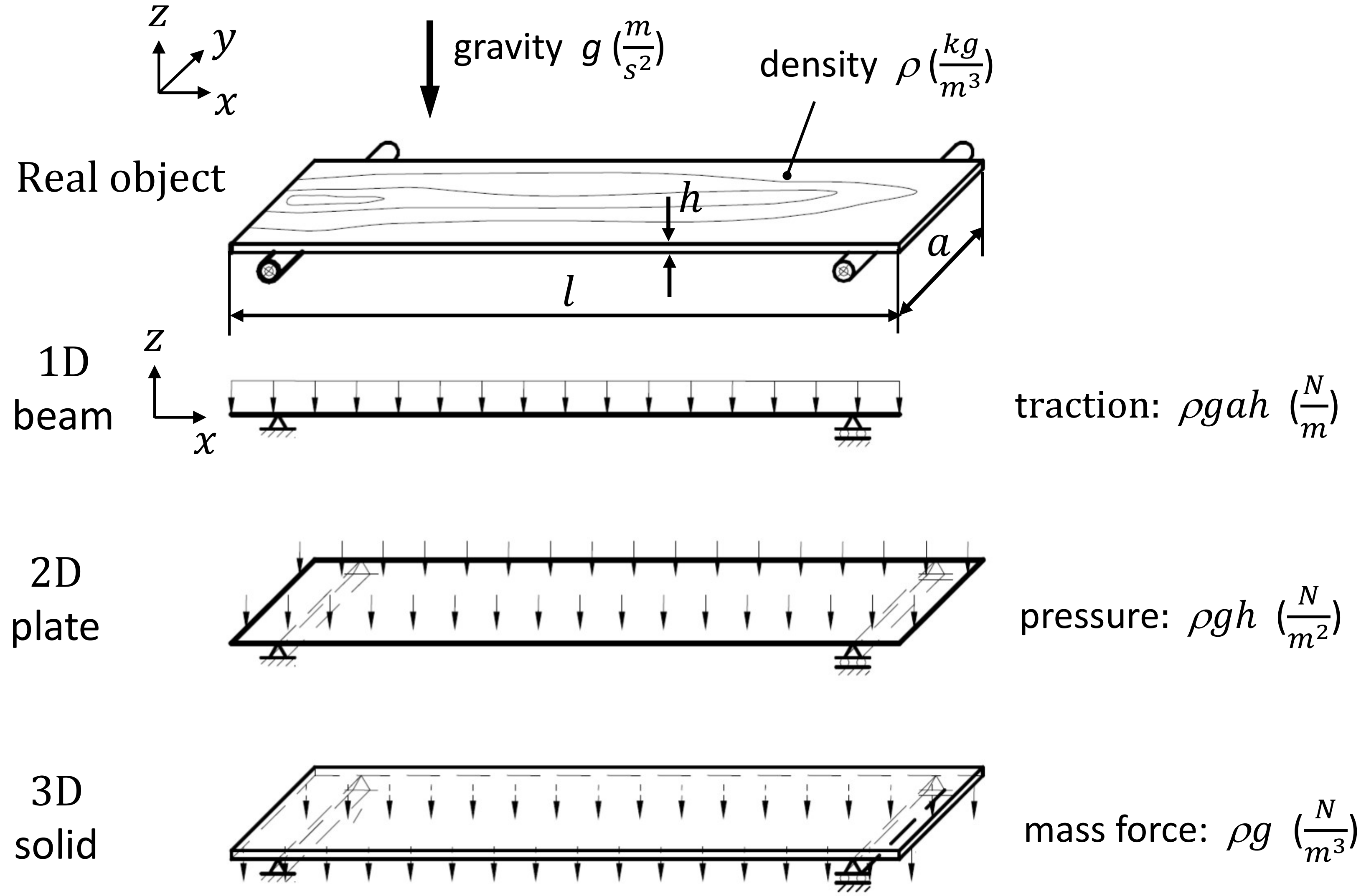
Errors

total error = modelling error + discretization error + numerical error

modelling error ≈ discretization error ≈ numerical error → min

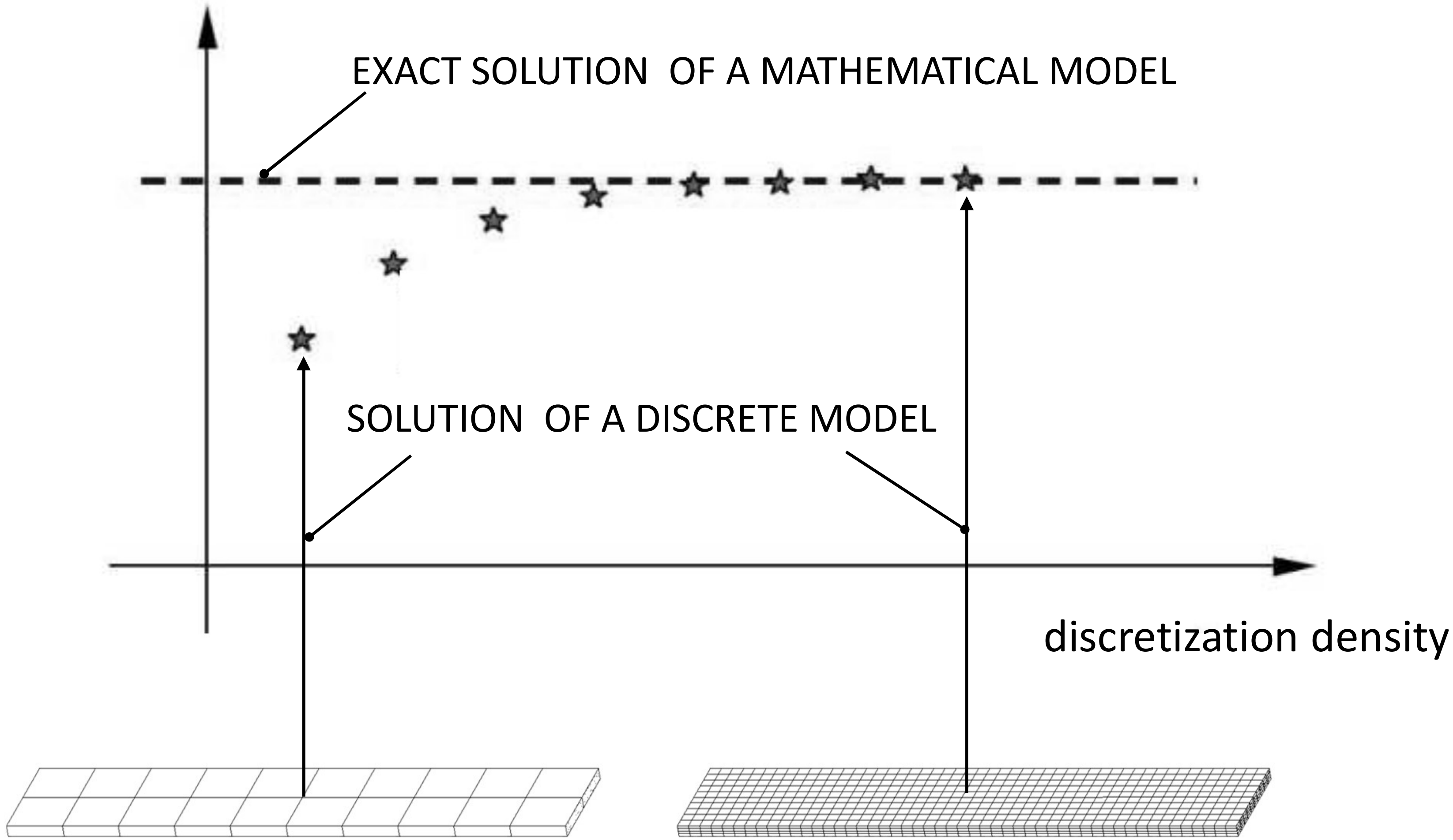


Example. Wooden board loaded by gravity



Example. Wooden board loaded by gravity

Discrete model



FE modeling – basic steps

Mathematical model

↓

Preprocessor

geometry, element types, material properties, approximation, discretization

↓

Discrete model

↓

Solution

analysis type, boundary conditions, solver, content of a result file

↓ *numerical calculations*

NUMERICAL RESULT

↓

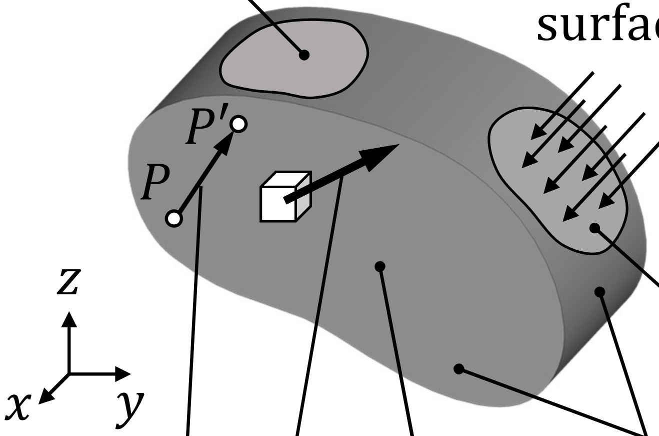
Postprocessor

numbers, graphs, contour maps, animations

Boundary value problem of solid body mechanics

boundary Γ_u ($u = u_0$)

surface load $\lfloor p \rfloor_{1\times 3} = \lfloor p_x, p_y, p_z \rfloor$



$$\Omega = \sum_{e=1}^{NOE} \Omega_e \text{ and } \Omega_i \underset{i \neq j}{\cap} \Omega_j = 0$$

boundary Γ_p

boundary Γ

domain Ω

mass forces $\lfloor X \rfloor_{1\times 3} = \lfloor X, Y, Z \rfloor = \rho \lfloor a_x, a_y, a_z \rfloor$

accelerations

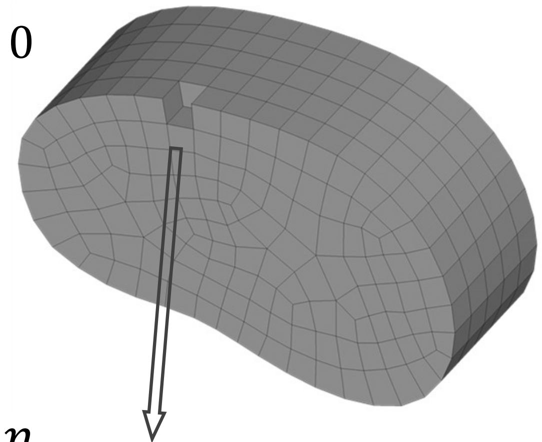
UNKNOWN FUNCTION

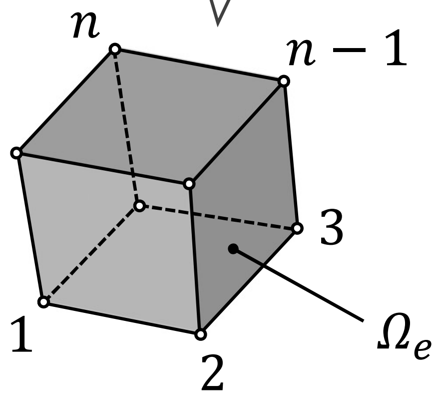
displacement vector $\{u\}_{3\times 1} = \begin{Bmatrix} u(x,y,z) \\ v(x,y,z) \\ w(x,y,z) \end{Bmatrix}$

FE model

NOE – no. of FEs

NON – no. of nodes





Finite element with n - nodes

Nodal approximation inside the finite element with n - nodes

displacement vector $\underset{3\times 1}{\{u\}} = \underset{3\times n_e}{[N(\xi,\eta,\zeta)]}\underset{n_e\times 1}{\{q\}_e}$

$\underset{3\times n_e}{[N(\xi,\eta,\zeta)]}$ − matrix of shape funtions

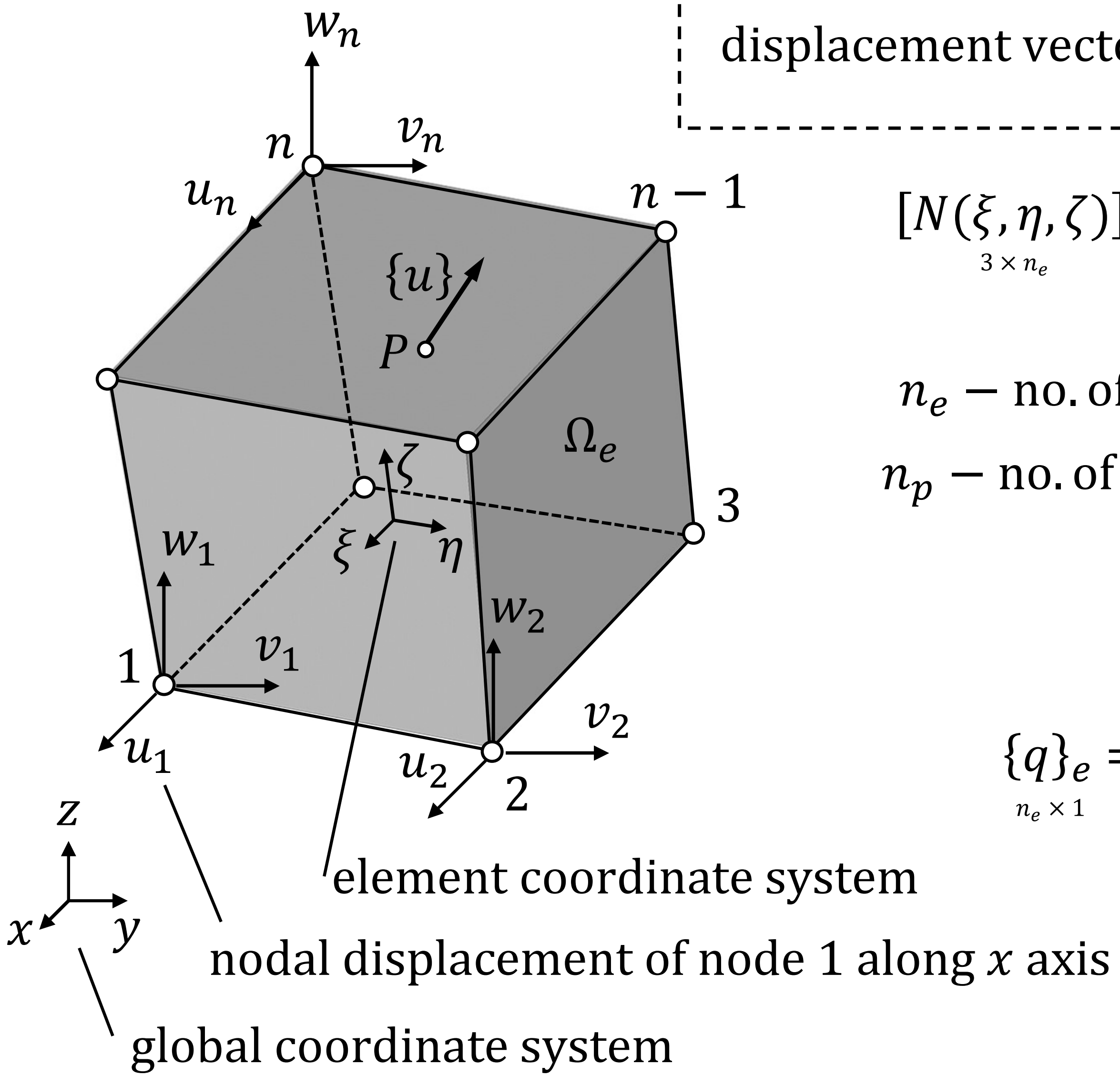
$$n_e = n \cdot n_p$$

n_e − no. of degrees of freedom in FE

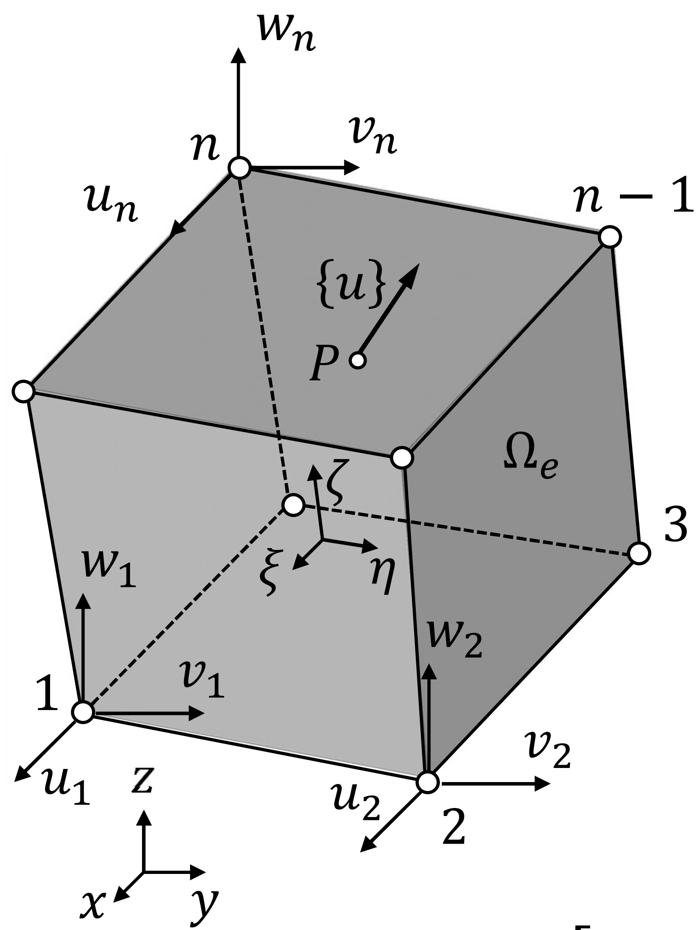
n_p − no. of degrees of freedom per node

$$\underset{n_e\times 1}{\{q\}_e} = \begin{Bmatrix} u_1 \\ v_1 \\ w_1 \\ \vdots \\ u_n \\ v_n \\ w_n \end{Bmatrix}_e$$

− local vector of nodal parameters



Matrix of shape functions



Nodal approximation:

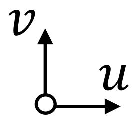
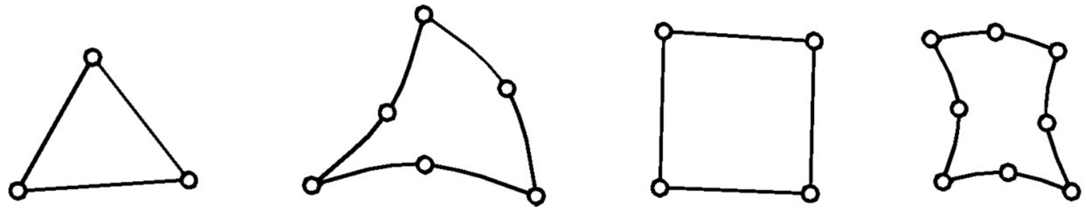
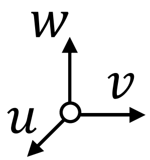
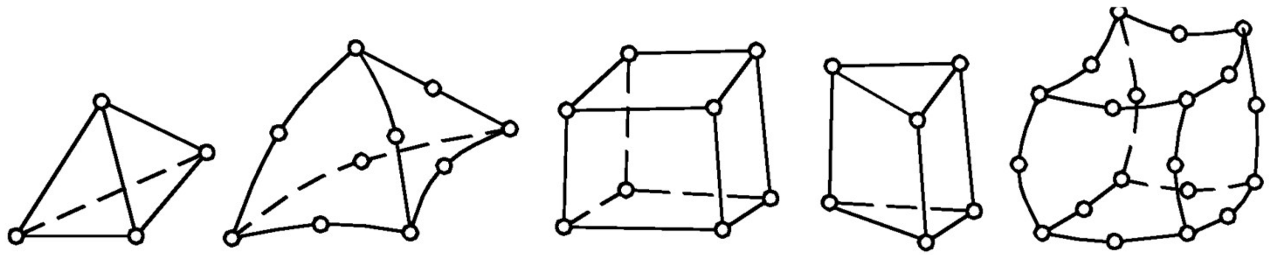
$$\underset{3\times 1}{\{u\}} = \underset{3\times n_e}{[N(\xi,\eta,\zeta)]}\underset{n_e\times 1}{\{q\}_e}$$

$$u = N_1 \cdot u_1 + N_2 \cdot u_2 + \ \ldots + N_n \cdot u_n$$
$$v = N_1 \cdot v_1 + N_2 \cdot v_2 + \ \ldots + N_n \cdot v_n$$
$$w = N_1 \cdot w_1 + N_2 \cdot w_2 + \ \ldots + N_n \cdot w_n$$

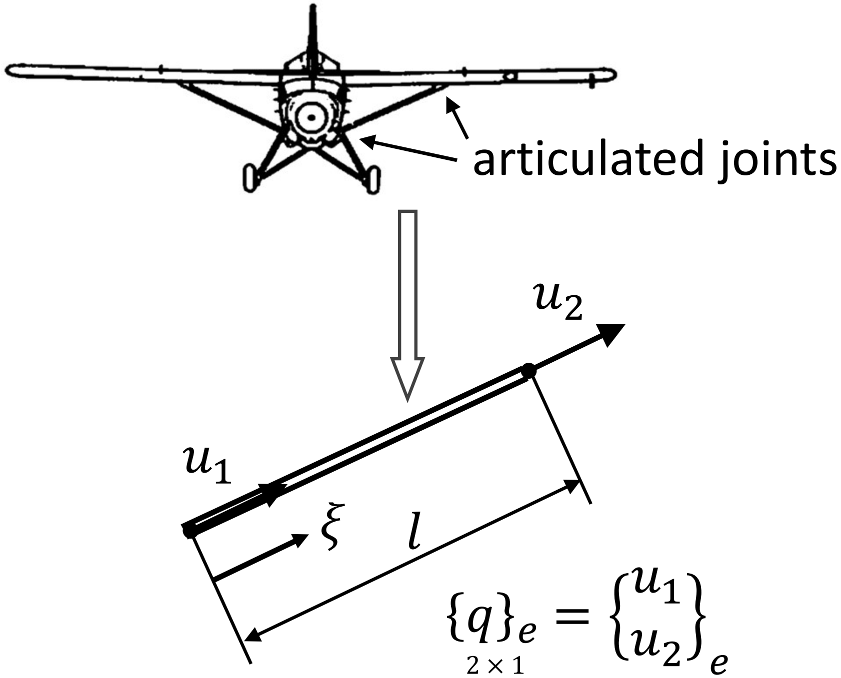
$$\underset{n_e\times 1}{\{q\}_e} = \begin{Bmatrix} u_1 \\ v_1 \\ w_1 \\ \vdots \\ u_n \\ v_n \\ w_n \end{Bmatrix}_e$$

$$\underset{3\times n_e}{[N(\xi,\eta,\zeta)]} = \begin{bmatrix} N_1 & 0 & 0 & N_2 & 0 & 0 & & N_n & 0 & 0 \\ 0 & N_1 & 0 & 0 & N_2 & 0 & \ldots & 0 & N_n & 0 \\ 0 & 0 & N_1 & 0 & 0 & N_2 & & 0 & 0 & N_n \end{bmatrix}$$

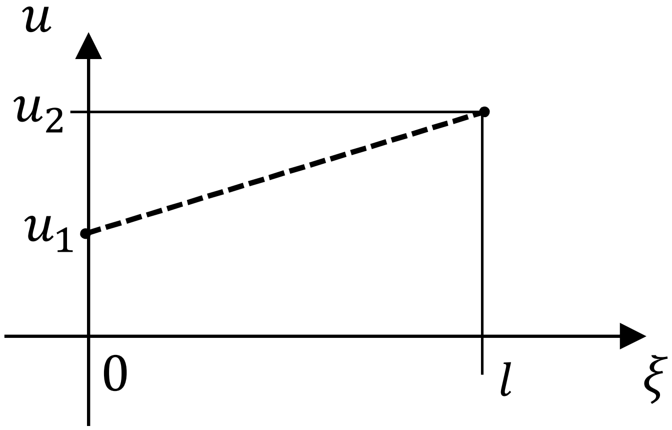
Examples of finite elements

Type	n_e − number of degrees of freedom in FE
rods	 2, 4, 6 6
2D 	 6 12 8 16
3D 	 12 30 24 18 60

Example: shape functions for a finite element representing a strut



$$\{q\}_{\underset{2\times 1}{e}} = \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}_e$$



linear function:

$$u(\xi) = \frac{u_2 - u_1}{l}\xi + u_1$$

$$u(\xi) = \frac{u_2 - u_1}{l}\xi + u_1 = \frac{u_2}{l}\xi - \frac{u_1}{l}\xi + u_1 = \left(1 - \frac{\xi}{l}\right)u_1 + \frac{\xi}{l}u_2 =$$

$$= N_1(\xi)\cdot u_1 + N_2(\xi)\cdot u_2 = \lfloor N_1, N_2 \rfloor \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}_e = \lfloor \underset{1\times 2}{N(\xi)} \rfloor \{q\}_{\underset{2\times 1}{e}}$$

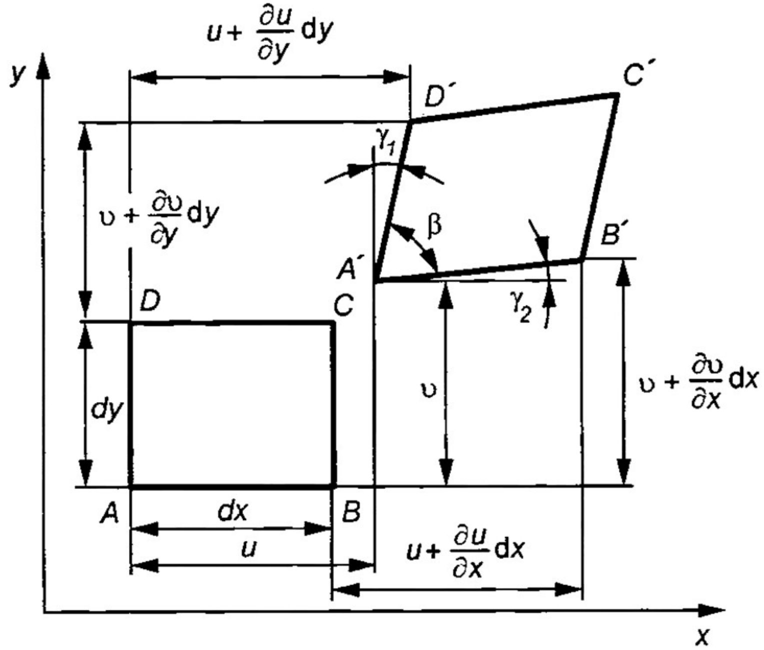
shape functions: $N_1(\xi) = 1 - \frac{\xi}{l}$; $N_2(\xi) = \frac{\xi}{l}$

Strain components

normal strains:

$$\varepsilon_x = \frac{(A'B')_x - AB}{AB} = \frac{(\mathrm{dx} + u + \frac{\partial u}{\partial x}\,\mathrm{dx} - u) - \mathrm{dx}}{\mathrm{dx}} = \frac{\partial u}{\partial x}$$

$$\varepsilon_y = \frac{\partial v}{\partial y} \quad ; \quad \varepsilon_z = \frac{\partial w}{\partial z}$$



shear strains:

$$\gamma_{xy} = \frac{\pi}{2} - \beta = \gamma_1 + \gamma_2$$

$$\gamma_1 \cong \tan\gamma_1 = \frac{(A'D')_x}{(A'D')_y} = \frac{u + \frac{\partial u}{\partial y}\,\mathrm{dy} - u}{\mathrm{dy} + v + \frac{\partial v}{\partial y}\,\mathrm{dy} - v} = \frac{\frac{\partial u}{\partial y}}{1 + \frac{\partial v}{\partial y}} = \frac{\frac{\partial u}{\partial y}}{1 + \varepsilon_y} = \frac{\partial u}{\partial y}$$

$$\gamma_2 \cong \frac{\partial v}{\partial x} \quad \rightarrow \quad \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$$

small deformations: $\varepsilon_y \ll 1$

$$\gamma_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \quad ; \quad \gamma_{zx} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} \quad ; \quad \gamma_{ij} = \gamma_{ji}$$

Strain tensor. Vector of strain components

strain tensor:

$$\underset{3\times 3}{\boldsymbol{\varepsilon}} = \begin{bmatrix} \varepsilon_x & \gamma_{xy}/2 & \gamma_{xz}/2 \\ \gamma_{yx}/2 & \varepsilon_y & \gamma_{yz}/2 \\ \gamma_{zx}/2 & \gamma_{zy}/2 & \varepsilon_z \end{bmatrix}$$

vector of strain components:

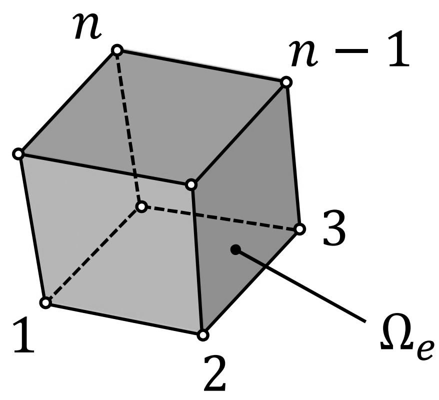
$$\underset{6\times 1}{\{\varepsilon\}} = \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{Bmatrix} = \begin{bmatrix} \frac{\partial}{\partial x} & 0 & 0 \\ 0 & \frac{\partial}{\partial y} & 0 \\ 0 & 0 & \frac{\partial}{\partial z} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial z} & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} & 0 & \frac{\partial}{\partial x} \end{bmatrix} \begin{Bmatrix} u \\ v \\ w \end{Bmatrix} = \underset{6\times 3}{[R]}\underset{3\times 1}{\{u\}} \quad ; \quad \underset{1\times 6}{\lfloor \varepsilon \rfloor} = \underset{1\times 3}{\lfloor u \rfloor}\underset{3\times 6}{[R]^T}$$

gradient matrix

Strain – displacement matrix of a finite element

nodal approximation in a finite element:

$$\underset{3\times 1}{\{u\}} = \underset{3\times n_e}{[N(\xi,\eta,\zeta)]}\underset{n_e\times 1}{\{q\}_e}$$



vector of strain components in a finite element:

$$\underset{6\times 1}{\{\varepsilon\}} = \underset{6\times 3}{[R]}\underset{3\times 1}{\{u\}} = \underset{6\times 3}{[R]}\underset{3\times n_e}{[N]}\underset{n_e\times 1}{\{q\}_e} = \underset{6\times n_e}{[B]}\underset{n_e\times 1}{\{q\}_e} \quad ; \quad \underset{1\times 6}{\lfloor\varepsilon\rfloor} = \underset{1\times n_e}{\lfloor q\rfloor_e}\underset{n_e\times 6}{[B]^T}$$

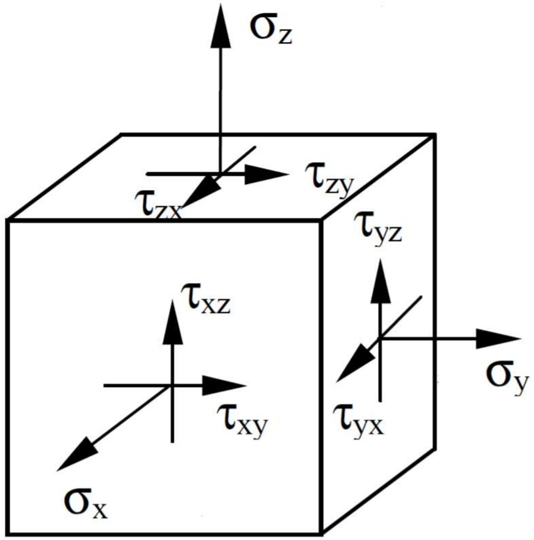
$$\underset{6\times n_e}{[B]} = \underset{6\times 3}{[R]}\underset{3\times n_e}{[N]} - \textit{strain–displacement matrix}$$

Stress components

normal stresses:

σ_x ; σ_y ; σ_z

positive value - tension, negative value - compression



shear stress components:

τ_{xy} ; τ_{yz} ; τ_{zx} ; $\tau_{ij} = \tau_{ji}$

equivalent stresses:

Von Mises stress:

$$\sigma_{EQV} = \sqrt{\frac{1}{2}\left(\left(\sigma_x - \sigma_y\right)^2 + \left(\sigma_y - \sigma_z\right)^2 + (\sigma_z - \sigma_x)^2\right) + 3({\tau_{xy}}^2 + {\tau_{yz}}^2 + {\tau_{zx}}^2)}$$

Tresca stress: $\sigma_{INT} = \sigma_1 - \sigma_3 = 2\tau_{max}$ ← *maximum shear stress*

the first principal stress (σ_1) *the third principal stress* (σ_3)

Stress tensor. Vector of stress components

stress tensor:

$$\underset{3\times 3}{\boldsymbol{\sigma}} = \begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z \end{bmatrix} \equiv \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix}$$

in coordinate system x, y, z — *in the principal coordinate system*

vector of stress components:

$$\underset{6\times 1}{\{\sigma\}} = \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{zx} \end{Bmatrix}$$

Constitutive matrix

linear isotropic material (Hooke's law):

$$\underset{6\times 1}{\{\sigma\}} = \underset{6\times 6}{[D]}\ \underset{6\times 1}{\{\varepsilon\}}$$

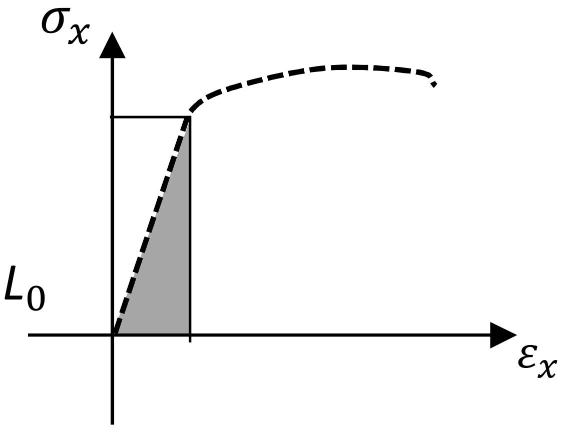
↑

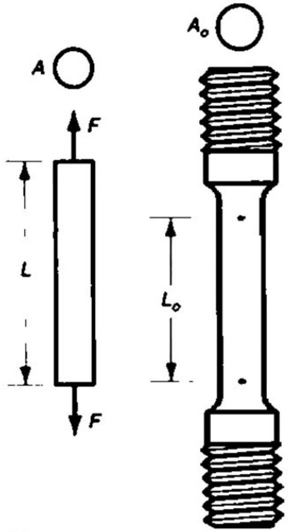
constitutive matrix

$$\underset{6\times 6}{[D]} = \frac{E}{(1+\nu)(1-2\nu)}\begin{bmatrix} 1-\nu & \nu & \nu & 0 & 0 & 0 \\ \nu & 1-\nu & \nu & 0 & 0 & 0 \\ \nu & \nu & 1-\nu & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.5-\nu & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.5-\nu & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.5-\nu \end{bmatrix}$$

E – Young's modulus, ν *– Poisson's ratio*

Example: uniaxial tensile test





$\sigma_x = \frac{F}{A_0}$; $\varepsilon_x = \frac{L-L_0}{L_0}$; $\varepsilon_y = \varepsilon_z = \varepsilon_T$

elastic strain energy: $U = \frac{1}{2}\sigma_x \; \varepsilon_x \; A_0 \; L_0$

$$\underset{6\times 1}{\{\sigma\}} = \underset{6\times 6}{[D]} \; \underset{6\times 1}{\{\varepsilon\}}$$

$$\begin{Bmatrix} \sigma_x \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & \nu & 0 & 0 & 0 \\ \nu & 1-\nu & \nu & 0 & 0 & 0 \\ \nu & \nu & 1-\nu & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.5-\nu & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.5-\nu & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.5-\nu \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_T \\ \varepsilon_T \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

2nd equation:

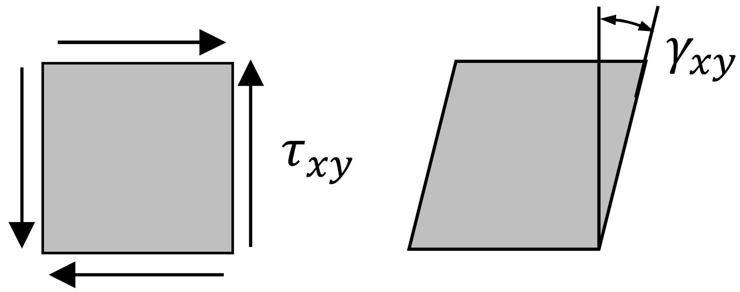
$$0 = \frac{E}{(1+\nu)(1-2\nu)}(\nu\varepsilon_x + (1-\nu)\varepsilon_T + \nu\varepsilon_T) \;\rightarrow\; \boxed{\varepsilon_T = -\nu\varepsilon_x}$$

1st equation:

$$\sigma_x = \frac{E}{(1+\nu)(1-2\nu)}((1-\nu)\varepsilon_x + \nu\varepsilon_T + \nu\varepsilon_T) = \frac{E}{(1-\nu-2\nu^2)}((1-\nu)\varepsilon_x - \nu^2\varepsilon_x - \nu^2\varepsilon_x) \;\rightarrow\; \boxed{\sigma_x = E\varepsilon_x}$$

Example: pure shear

τ_{xy} ; γ_{xy}



$$\underset{6\times 1}{\{\sigma\}} = \underset{6\times 6}{[D]}\ \underset{6\times 1}{\{\varepsilon\}}$$

$$\begin{Bmatrix} 0 \\ 0 \\ 0 \\ \tau_{xy} \\ 0 \\ 0 \end{Bmatrix} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & \nu & 0 & 0 & 0 \\ \nu & 1-\nu & \nu & 0 & 0 & 0 \\ \nu & \nu & 1-\nu & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.5-\nu & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.5-\nu & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.5-\nu \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ 0 \\ \gamma_{xy} \\ 0 \\ 0 \end{Bmatrix}$$

4th equation:

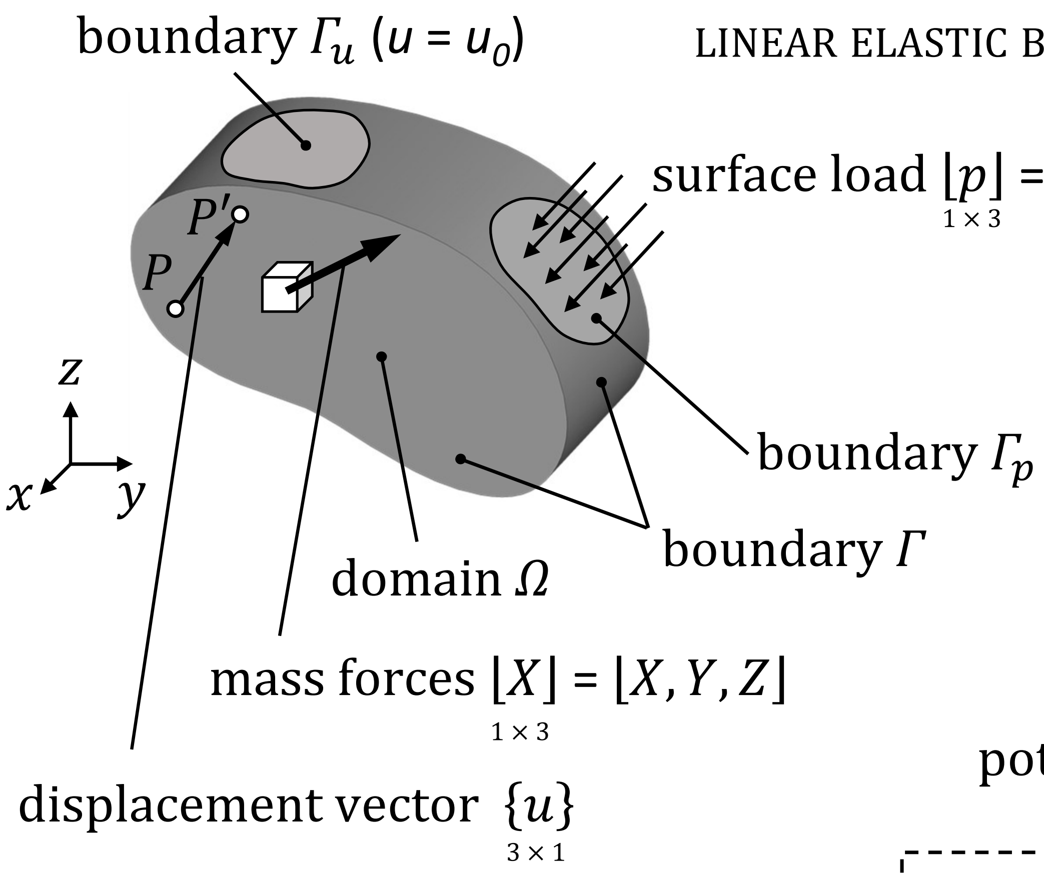
$$\tau_{xy} = \frac{E}{(1+\nu)(1-2\nu)}(0.5-\nu)\gamma_{xy} = \frac{E}{2(1+\nu)(0.5-\nu)}(0.5-\nu)\gamma_{xy} = \frac{E}{2(1+\nu)}\gamma_{xy} \rightarrow$$

$$\boxed{\tau_{xy} = G\,\gamma_{xy}}$$

$G = \frac{E}{2(1+\nu)}$ – *Kirchof's modulus (shear modulus)*

Elastic strain energy. Potential energy of loading

LINEAR ELASTIC BODY



elastic strain energy:

$$U = \frac{1}{2}\int_{\Omega} \underset{1\times 6}{\lfloor \varepsilon \rfloor} \underset{6\times 1}{\{\sigma\}}\, d\Omega$$

potential energy of loading:

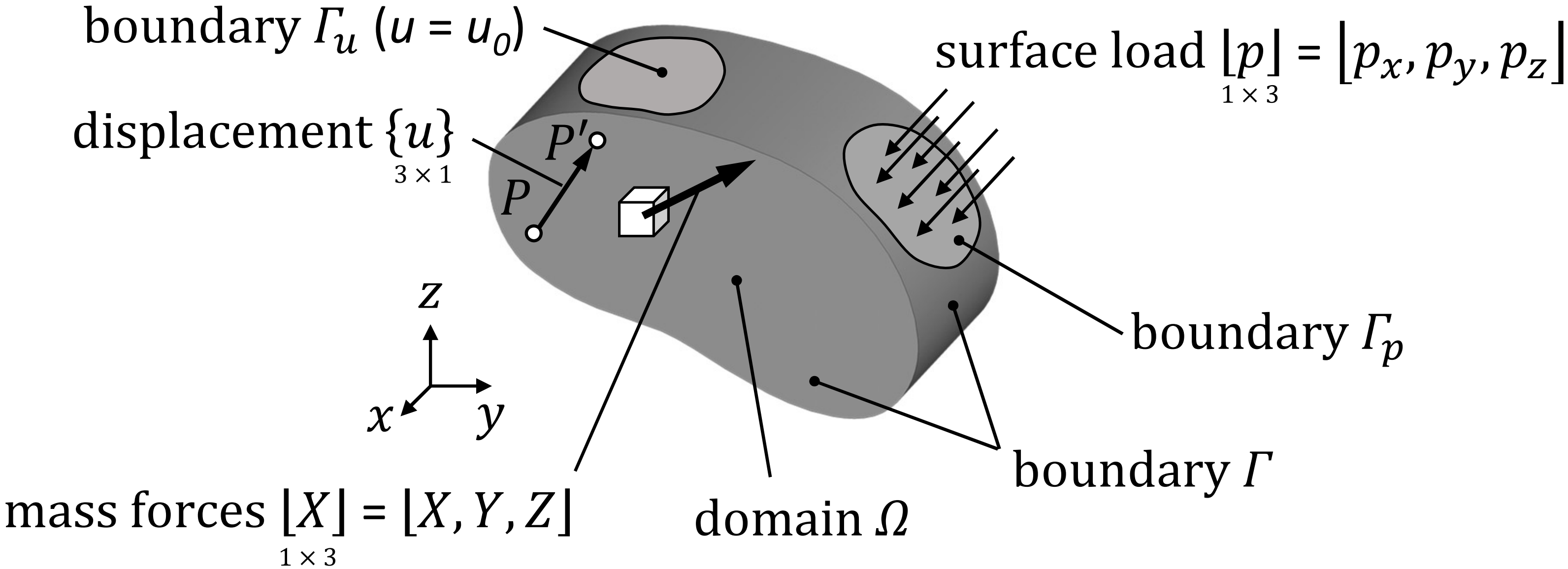
$$W = \int_{\Omega} \underset{1\times 3}{\lfloor X \rfloor} \underset{3\times 1}{\{u\}}\, d\Omega + \int_{\Gamma_p} \underset{1\times 3}{\lfloor p \rfloor} \underset{3\times 1}{\{u\}}\, d\Gamma_p$$

Minimum total potential energy principle

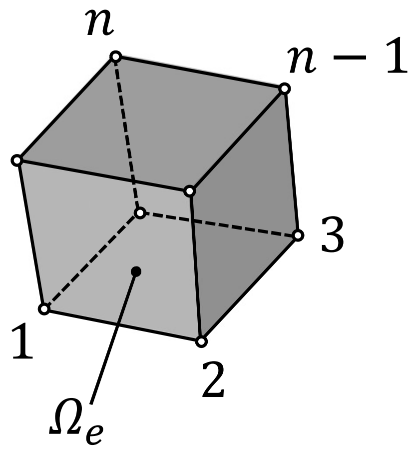
total potential energy: $V = U - W$

The displacement field $\{u\}$ that represents solution of the problem fulfils displacement boundary conditons on Γ_u and minimizes the total potential energy V.

$$V \rightarrow min$$



Elastic strain energy in a finite element. Local stiffness matrix



$\underset{n_e \times 1}{\{q\}_e}$ - local vector of nodal parameters

elastic strain energy in a finite element:

$$U_e = \frac{1}{2}\int_{\Omega_e} \underset{1\times 6}{\lfloor \varepsilon \rfloor} \underset{6\times 1}{\{\sigma\}}\, d\Omega_e = \frac{1}{2} \underset{1\times n_e}{\lfloor q \rfloor_e} \int_{\Omega_e} \underset{n_e\times 6}{[B]^T} \underset{6\times 6}{[D]} \underset{6\times n_e}{[B]}\, d\Omega_e\, \underset{n_e\times 1}{\{q\}_e} = \frac{1}{2} \underset{1\times n_e}{\lfloor q \rfloor_e} \underset{n_e\times n_e}{[k]_e} \underset{n_e\times 1}{\{q\}_e}$$

$$\underset{6\times 1}{\{\sigma\}} = \underset{6\times 6}{[D]}\ \underset{6\times 1}{\{\varepsilon\}}$$

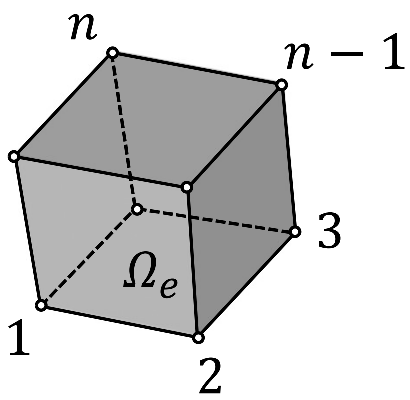
$$\underset{1\times 6}{\lfloor \varepsilon \rfloor} = \underset{1\times n_e}{\lfloor q \rfloor_e} \underset{n_e\times 6}{[B]^T} \qquad \underset{6\times 1}{\{\varepsilon\}} = \underset{6\times n_e}{[B]} \underset{n_e\times 1}{\{q\}_e}$$

local stiffness matrix:

$$\underset{n_e\times n_e}{[k]_e} = \int_{\Omega_e} \underset{n_e\times 6}{[B]^T} \underset{6\times 6}{[D]} \underset{6\times n_e}{[B]} d\Omega_e$$

Elastic strain energy in a finite element

local notation:



n – no. of nodes per FE

n_p – no. of nodal parameters per node

no. of degrees of freedom in FE:

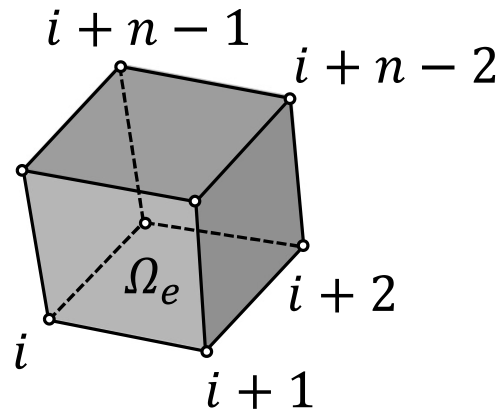
$$n_e = n \cdot n_p$$

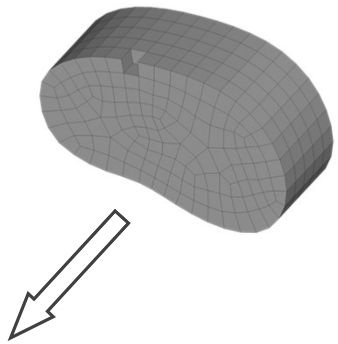
$\underset{n_e \times 1}{\{q\}_e}$ - local vector of nodal parameters

$$U_e = \frac{1}{2} \underset{1 \times n_e}{\lfloor q \rfloor_e} \underset{n_e \times n_e}{[k]_e} \underset{n_e \times 1}{\{q\}_e}$$

↑ local stiffness matrix

global notation:





NON – no. of nodes

n_p – no. of nodal parameters per node

no. of degrees of freedom:

$$NDOF = NON \cdot n_p$$

$\underset{NDOF \times 1}{\{q\}}$ - global vector of nodal parameters

$$U_e = \frac{1}{2} \cdot \underset{1 \times NDOF}{\lfloor q \rfloor} \cdot \underset{NDOF \times NDOF}{[k]_e^*} \cdot \underset{NDOF \times 1}{\{q\}}$$

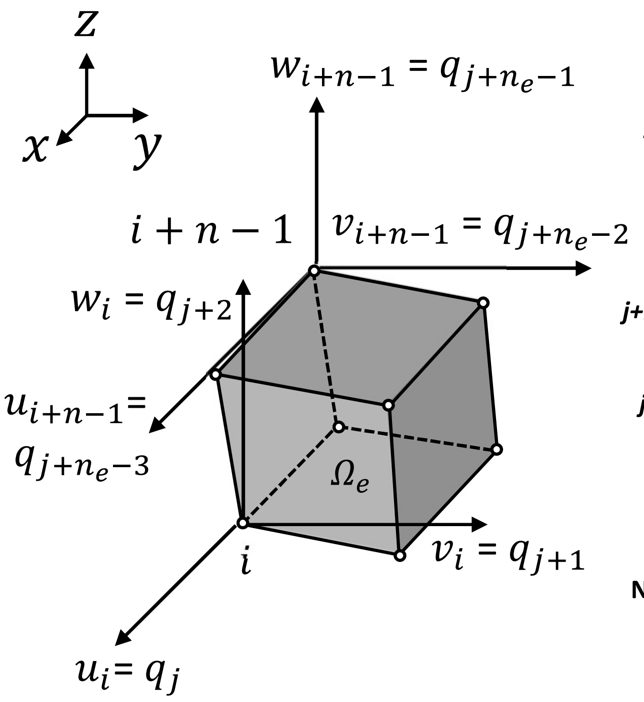
↑ extended local stiffness matrix

Extended local stiffness matrix of a finite element

$$\underset{NDOF \times 1}{\{q\}} = \begin{Bmatrix} q_1 \\ q_2 \\ \vdots \\ q_j \\ \vdots \\ q_{NDOF} \end{Bmatrix}$$

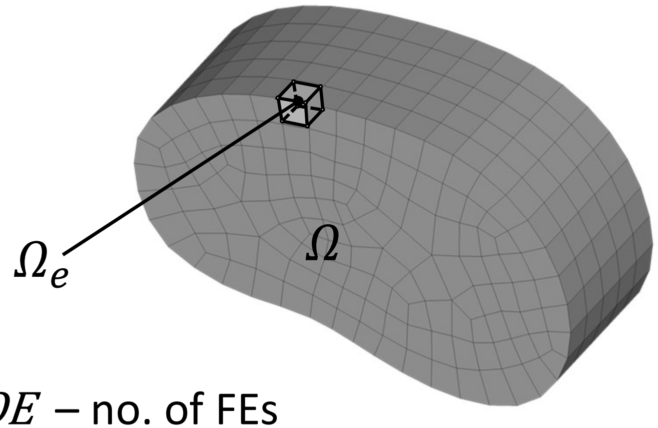
$$[k]_e^* =$$

	1	2	…	j–1	j	j+1	…	$j+n_e-1$	$j+n_e$	…	NDOF
1	0	0	…	0	0	0	…	0	0	…	0
2	0	0	…	0	0	0	…	0	0	…	0
…	…	…	…	0	0	0	…	0	0	…	0
j–1	0	0	0	0	0	0	…	0	0	…	0
j	0	0	0	0	k_{11}	k_{12}	…	k_{1n_e}	0	…	0
j+1	0	0	0	0	k_{21}	k_{22}	…	k_{2n_e}	0	…	0
…	…	…	…	…	…	…	…	…	0	…	0
$j+n_e-1$	0	0	0	0	k_{n_e1}	k_{n_e2}	…	$k_{n_en_e}$	0	…	0
$j+n_e$	0	0	0	0	0	0	0	0	0	…	0
…	…	…	…	…	…	…	…	…	…	…	0
NDOF	0	0	0	0	0	0	0	0	0	0	0



(assumed ascending order of components)

Elastic strain energy in a FE model. Global stiffness matrix



$$\Omega = \sum_{e=1}^{NOE} \Omega_e \quad \rightarrow \quad \boxed{U = \sum_{e=1}^{NOE} U_e}$$

NOE – no. of FEs

$NDOF$ – no. of degrees of freedom

$\underset{NDOF \times 1}{\{q\}}$ - global vector of nodal parameters

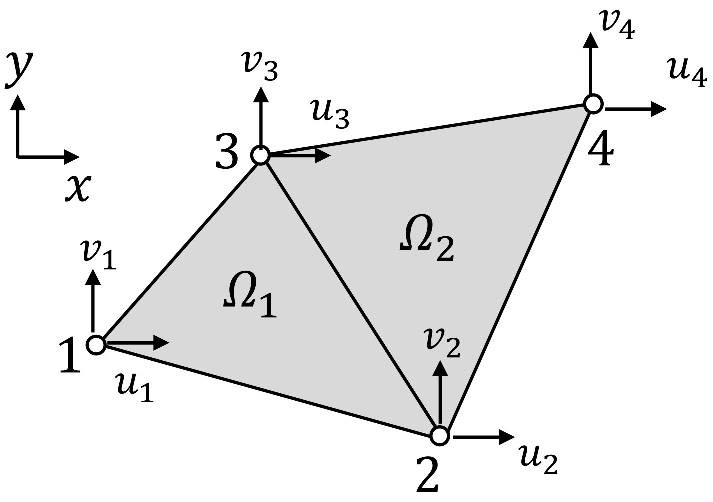
elastic strain energy in a finite element model:

$$U = \sum_{e=1}^{NOE} U_e = \sum_{e=1}^{NOE} \frac{1}{2} \cdot \underset{1 \times NDOF}{\lfloor q \rfloor} \cdot \underset{NDOF \times NDOF}{[k]_e^*} \cdot \underset{NDOF \times 1}{\{q\}} = \frac{1}{2} \underset{1 \times NDOF}{\lfloor q \rfloor} \cdot \sum_{e=1}^{NOE} \underset{NDOF \times NDOF}{[k]_e^*} \cdot \underset{NDOF \times 1}{\{q\}} =$$

$$= \frac{1}{2} \cdot \underset{1 \times NDOF}{\lfloor q \rfloor} \cdot \underset{NDOF \times NDOF}{[K]} \cdot \underset{NDOF \times 1}{\{q\}}$$

global stiffness matrix: $\boxed{\underset{NDOF \times NDOF}{[K]} = \sum_{e=1}^{NOE} [k]_e^*}$

Example: global stiffness matrix of a 2D model with two 3-node triangles



global notation:

$NOE = 2$

$NON = 4$

$n = 3$

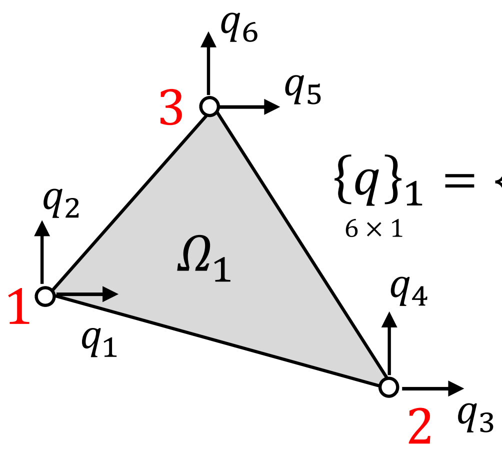
$n_p = 2$; (u, v)

$n_e = n \cdot n_p = 6$

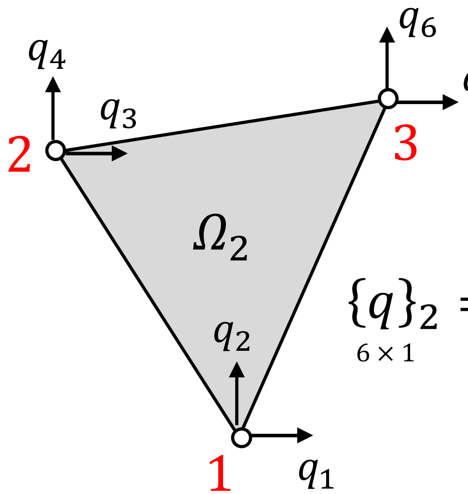
$NDOF = NON \cdot n_p = 8$

$$\underset{8\times 1}{\{q\}} = \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \\ q_6 \\ q_7 \\ q_8 \end{Bmatrix} = \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{Bmatrix}$$

local notation:



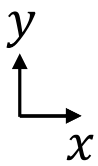
$$\underset{6\times 1}{\{q\}_1} = \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \\ q_6 \end{Bmatrix}_1 = \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{Bmatrix}_1$$

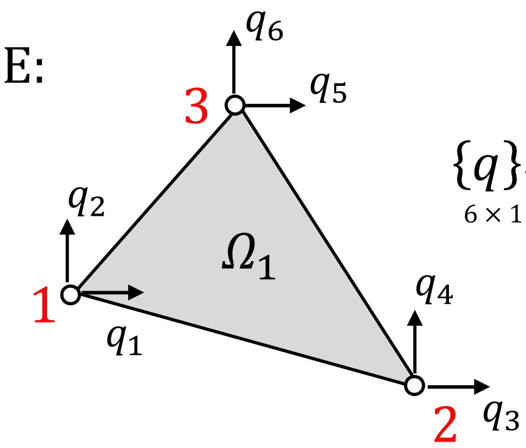


$$\underset{6\times 1}{\{q\}_2} = \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \\ q_6 \end{Bmatrix}_2 = \begin{Bmatrix} u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{Bmatrix}_2$$

Example: global stiffness matrix of a 2D model with two 3-node triangles

1st FE:





$$\underset{6\times 1}{\{q\}_1} = \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \\ q_6 \end{Bmatrix}_1 = \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{Bmatrix}_1$$

$$\underset{8\times 1}{\{q\}} = \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{Bmatrix}$$

$\underset{6\times 6}{[k]_1} =$

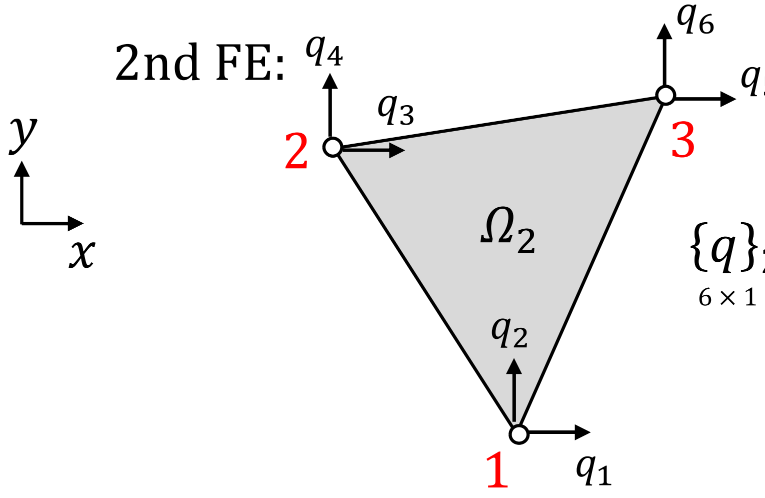
	1	2	3	4	5	6
1	a_1	b_1	c_1	d_1	e_1	f_1
2	b_1	g_1	h_1	i_1	j_1	k_1
3	c_1	h_1	l_1	m_1	n_1	o_1
4	d_1	i_1	m_1	p_1	r_1	s_1
5	e_1	j_1	n_1	r_1	t_1	$\bar{u}_1$
6	f_1	k_1	o_1	s_1	$\bar{u}_1$	$\bar{w}_1$

$\underset{8\times 8}{[k]_1^*} =$

	1	2	3	4	5	6	7	8
1	a_1	b_1	c_1	d_1	e_1	f_1	0	0
2	b_1	g_1	h_1	i_1	j_1	k_1	0	0
3	c_1	h_1	l_1	m_1	n_1	o_1	0	0
4	d_1	i_1	m_1	p_1	r_1	s_1	0	0
5	e_1	j_1	n_1	r_1	t_1	$\bar{u}_1$	0	0
6	f_1	k_1	o_1	s_1	$\bar{u}_1$	$\bar{w}_1$	0	0
7	0	0	0	0	0	0	0	0
8	0	0	0	0	0	0	0	0

Example: global stiffness matrix of a 2D model with two 3-node triangles

2nd FE:



$$\underset{6\times 1}{\{q\}_2} = \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \\ q_6 \end{Bmatrix}_2 = \begin{Bmatrix} u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{Bmatrix}_2$$

$$\underset{8\times 1}{\{q\}} = \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{Bmatrix}$$

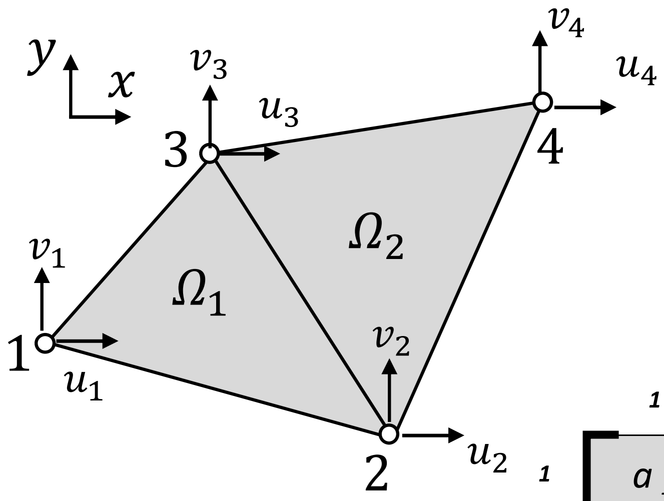
$\underset{6\times 6}{[k]_2} =$

	1	2	3	4	5	6
1	a_2	b_2	c_2	d_2	e_2	f_2
2	b_2	g_2	h_2	i_2	j_2	k_2
3	c_2	h_2	l_2	m_2	n_2	o_2
4	d_2	i_2	m_2	p_2	r_2	s_2
5	e_2	j_2	n_2	r_2	t_2	$\bar{u}_2$
6	f_2	k_2	o_2	s_2	$\bar{u}_2$	$\bar{w}_2$

$\underset{8\times 8}{[k]_2^*} =$

	1	2	3	4	5	6	7	8
1	0	0	0	0	0	0	0	0
2	0	0	0	0	0	0	0	0
3	0	0	a_2	b_2	c_2	d_2	e_2	f_2
4	0	0	b_2	g_2	h_2	i_2	j_2	k_2
5	0	0	c_2	h_2	l_2	m_2	n_2	o_2
6	0	0	d_2	i_2	m_2	p_2	r_2	s_2
7	0	0	e_2	j_2	n_2	r_2	t_2	$\bar{u}_2$
8	0	0	f_2	k_2	o_2	s_2	$\bar{u}_2$	$\bar{w}_2$

Example: global stiffness matrix of a 2D model with two 3-node triangles



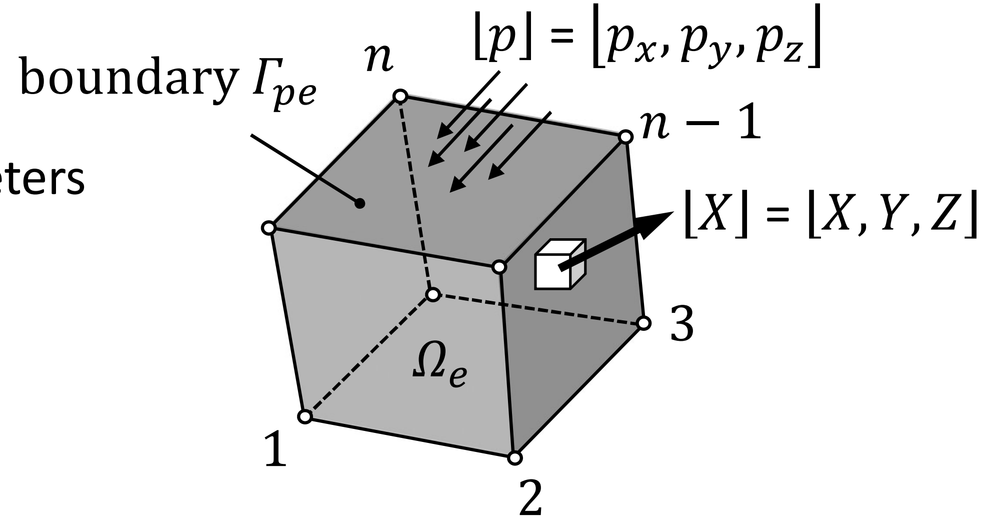
$$\underset{8\times 1}{\{q\}} = \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{Bmatrix}$$

$$\underset{8\times 8}{[K]} = \underset{8\times 8}{[k]_1^*} + \underset{8\times 8}{[k]_2^*} =$$

	1	2	3	4	5	6	7	8
1	a_1	b_1	c_1	d_1	e_1	f_1	0	0
2	b_1	g_1	h_1	i_1	j_1	k_1	0	0
3	c_1	h_1	l_1+a_2	m_1+b_2	n_1+c_2	o_1+d_2	e_2	f_2
4	d_1	i_1	m_1+b_2	p_1+g_2	r_1+h_2	s_1+i_2	j_2	k_2
5	e_1	j_1	n_1+c_2	r_1+h_2	t_1+l_2	$\bar{u}_1+m_2$	n_2	o_2
6	f_1	k_1	o_1+d_2	t_1+l_2	$\bar{u}_1+m_2$	$\bar{w}_1+p_2$	r_2	s_2
7	0	0	e_2	j_2	n_2	r_2	t_2	$\bar{u}_2$
8	0	0	f_2	k_2	o_2	s_2	$\bar{u}_2$	$\bar{w}_2$

Potential energy of loading in a finite element

$\underset{n_e \times 1}{\{q\}_e}$ - local vector of nodal parameters



potential energy of loading
in a finite element:

$$W_e = \int_{\Omega_e} \underset{1\times 3}{\lfloor X \rfloor}\underset{3\times 1}{\{u\}}\, d\Omega_e + \int_{\Gamma_{pe}} \underset{1\times 3}{\lfloor p \rfloor}\underset{3\times 1}{\{u\}}\, d\Gamma_{pe} = \int_{\Omega_e} \underset{1\times 3}{\lfloor X \rfloor}\underset{3\times n_e}{[N]}\underset{n_e\times 1}{\{q\}_e}\, d\Omega_e + \int_{\Gamma_{pe}} \underset{1\times 3}{\lfloor p \rfloor}\underset{3\times n_e}{[N]}\underset{n_e\times 1}{\{q\}_e}\, d\Gamma_{pe} =$$

$$\underset{3\times 1}{\{u\}} = \underset{3\times n_e}{[N]}\underset{n_e\times 1}{\{q\}_e}$$

$$= \left(\int_{\Omega_e} \underset{1\times 3}{\lfloor X \rfloor}\underset{3\times n_e}{[N]}\, d\Omega_e + \int_{\Gamma_{pe}} \underset{1\times 3}{\lfloor p \rfloor}\underset{3\times n_e}{[N]}\, d\Gamma_{pe}\right) \underset{n_e\times 1}{\{q\}_e} = \left(\underset{1\times n_e}{\lfloor F^X \rfloor_e} + \underset{1\times n_e}{\lfloor F^p \rfloor_e}\right)\underset{n_e\times 1}{\{q\}_e} = \underset{1\times n_e}{\lfloor F \rfloor_e}\underset{n_e\times 1}{\{q\}_e}$$

equivalent load vector: $$\underset{1\times n_e}{\lfloor F \rfloor_e} = \underset{1\times n_e}{\lfloor F^X \rfloor_e} + \underset{1\times n_e}{\lfloor F^p \rfloor_e}$$

Equivalent load vector

$$\underset{1\times n_e}{\lfloor F \rfloor_e} = \underset{1\times n_e}{\lfloor F^X \rfloor_e} + \underset{1\times n_e}{\lfloor F^p \rfloor_e}$$

equivalent load vector due to mass forces:

$$\underset{1\times n_e}{\lfloor F^X \rfloor_e} = \int_{\Omega_e} \underset{1\times 3}{\lfloor X \rfloor} \underset{3\times n_e}{[N]} d\Omega_e =$$

$$= \int_{\Omega_e} \lfloor X, Y, Z \rfloor \begin{bmatrix} N_1 & 0 & 0 & N_2 & 0 & 0 & & N_n & 0 & 0 \\ 0 & N_1 & 0 & 0 & N_2 & 0 & \dots & 0 & N_n & 0 \\ 0 & 0 & N_1 & 0 & 0 & N_2 & & 0 & 0 & N_n \end{bmatrix} d\Omega_e$$

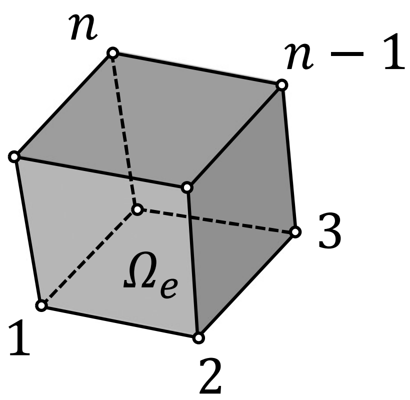
equivalent load vector due to surface load:

$$\underset{1\times n_e}{\lfloor F^p \rfloor_e} = \int_{\Gamma_{pe}} \underset{1\times 3}{\lfloor p \rfloor} \underset{3\times n_e}{[N]} d\Gamma_{pe} =$$

$$= \int_{\Gamma_{pe}} \lfloor p_x, p_y, p_z \rfloor \begin{bmatrix} N_1 & 0 & 0 & N_2 & 0 & 0 & & N_n & 0 & 0 \\ 0 & N_1 & 0 & 0 & N_2 & 0 & \dots & 0 & N_n & 0 \\ 0 & 0 & N_1 & 0 & 0 & N_2 & & 0 & 0 & N_n \end{bmatrix} d\Gamma_{pe}$$

Potential energy of loading in a finite element

local notation:



n – no. of nodes per FE

n_p – no. of nodal parameters per node

no. of degrees of freedom in FE:

$$n_e = n \cdot n_p$$

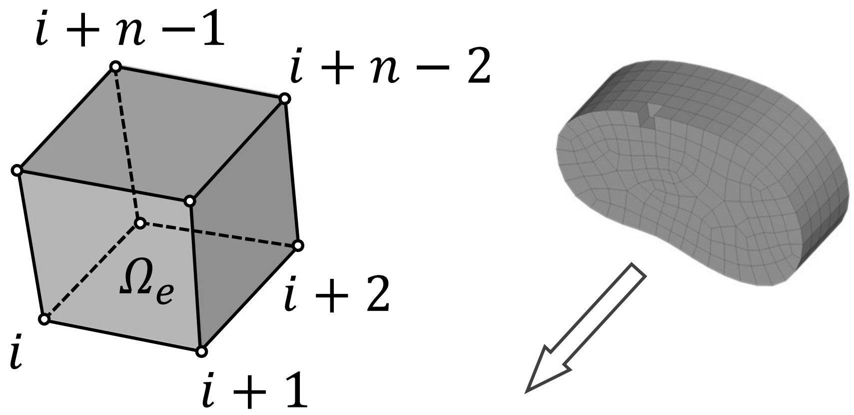
$\underset{n_e \times 1}{\{q\}_e}$ - local vector of nodal parameters

$$W_e = \underset{1 \times n_e}{\lfloor q \rfloor_e} \underset{n_e \times 1}{\{F\}_e}$$

↑

equivalent load vector

global notation:



NON – no. of nodes

n_p – no. of nodal parameters per node

no. of degrees of freedom:

$$NDOF = NON \cdot n_p$$

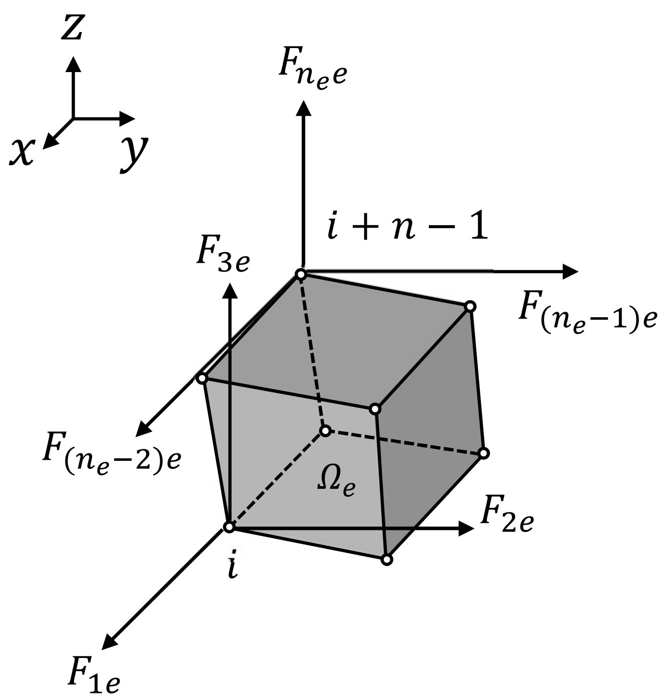
$\underset{NDOF \times 1}{\{q\}}$ - global vector of nodal parameters

$$W_e = \underset{1 \times NDOF}{\lfloor q \rfloor} \cdot \underset{NDOF \times 1}{\{F\}_e^*}$$

↑

extended equivalent load vector

Extended equivalent load vector in a finite element



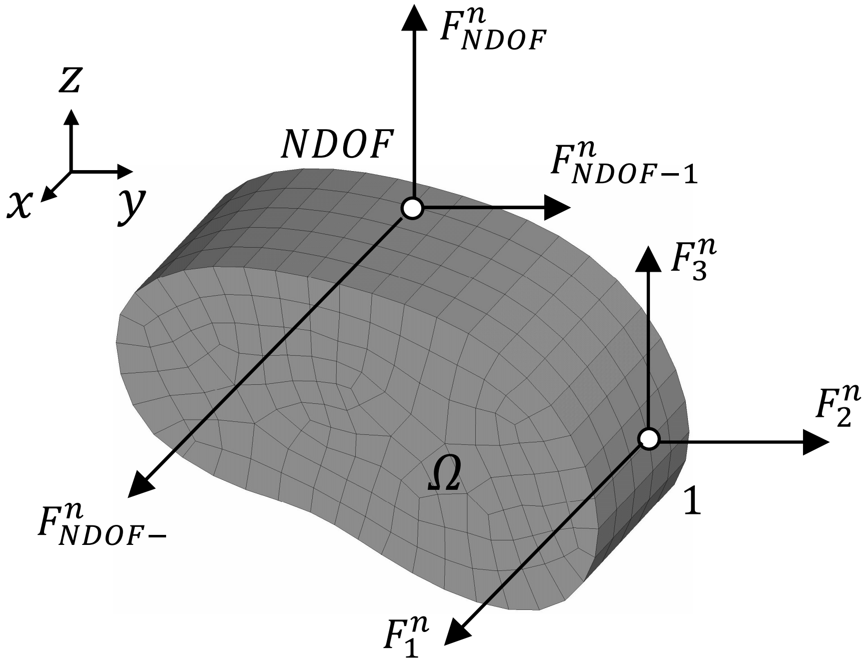
extended equivalent load vector:

$$\underset{NDOF\times 1}{\{F\}_e^*} = \begin{Bmatrix} 0 \\ 0 \\ \cdots \\ 0 \\ F_{1e} \\ F_{2e} \\ \cdots \\ F_{n_e e} \\ 0 \\ \cdots \\ 0 \end{Bmatrix} \quad \begin{matrix} 1 \\ 2 \\ \\ j-1 \\ j \\ j+1 \\ \\ j+n_e-1 \\ j+n_e \\ \\ NDOF \end{matrix}$$

(assumed ascending order of components)

equivalent load vector: $$\underset{n_e\times 1}{\{F\}_e} = \begin{Bmatrix} F_{1e} \\ F_{2e} \\ F_{3e} \\ \cdots \\ F_{(n_e-2)e} \\ F_{(n_e-1)e} \\ F_{n_e e} \end{Bmatrix}$$

Forces applied directly on nodes. Potential energy of nodal loads



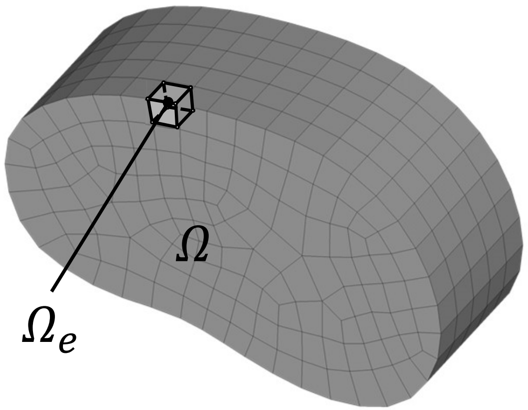
nodal load vector:

$$\underset{NDOF\times 1}{\{F\}^n} = \begin{Bmatrix} F^n_1 \\ F^n_2 \\ F^n_3 \\ \dots \\ F^n_{NDOF-2} \\ F^n_{NDO} \\ F^n_{NDOF} \end{Bmatrix}$$

potential energy of nodal loads:

$$W^n = \underset{1\times NDOF}{\lfloor q \rfloor} \cdot \underset{NDOF\times 1}{\{F\}^n}$$

Potential energy of loading in a FE model. Global load vector



NOE – no. of FEs
NDOF – no. of degrees of freedom

potential energy of element loads:

$$\Omega = \sum_{e=1}^{NOE} \Omega_e \quad \rightarrow \quad \boxed{W^e = \sum_{e=1}^{NOE} W_e}$$

potential energy of loading in a finite element model: $\boxed{W = W^e + W^n}$

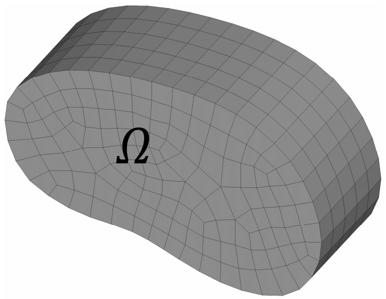
$$W = \sum_{e=1}^{NOE} W_e + W^n = \sum_{e=1}^{NOE} \underset{1\times NDOF}{\lfloor q \rfloor} \cdot \underset{NDOF\times 1}{\{F\}_e^*} + \underset{1\times NDOF}{\lfloor q \rfloor} \cdot \underset{NDOF\times 1}{\{F\}^n} = \underset{1\times NDOF}{\lfloor q \rfloor} \cdot \left(\sum_{e=1}^{NOE} \underset{NDOF\times 1}{\{F\}_e^*} + \underset{NDOF\times 1}{\{F\}^n}\right)$$

$$= \underset{1\times NDOF}{\lfloor q \rfloor} \cdot \left(\underset{NDOF\times 1}{\{F\}^e} + \underset{NDOF\times 1}{\{F\}^n}\right) \quad \rightarrow \quad W = \underset{1\times NDOF}{\lfloor q \rfloor} \cdot \underset{NDOF\times 1}{\{F\}}$$

global load vector of element loads

global load vector: $\boxed{\underset{NDOF\times 1}{\{F\}} = \underset{NDOF\times 1}{\{F\}^e} + \underset{NDOF\times 1}{\{F\}^n}}$

Total potential energy in a FE model. Set of linear equations



NOE – no. of FEs
$NDOF$ – no. of degrees of freedom

Total potential energy of the entire model:

$$V = U - W = \frac{1}{2} \cdot \underset{1 \times NDOF}{\lfloor q \rfloor} \cdot \underset{NDOF \times NDOF}{[K]} \cdot \underset{NDOF \times 1}{\{q\}} - \underset{1 \times NDOF}{\lfloor q \rfloor} \cdot \underset{NDOF \times 1}{\{F\}}$$

$$\underset{NDOF \times 1}{\{q\}} = ?$$

$$V \rightarrow min$$

$$\frac{\partial V}{\partial q_j} = 0 \rightarrow \boxed{\underset{NDOF \times NDOF}{[K]} \cdot \underset{NDOF \times 1}{\{q\}} = \underset{NDOF \times 1}{\{F\}}}$$

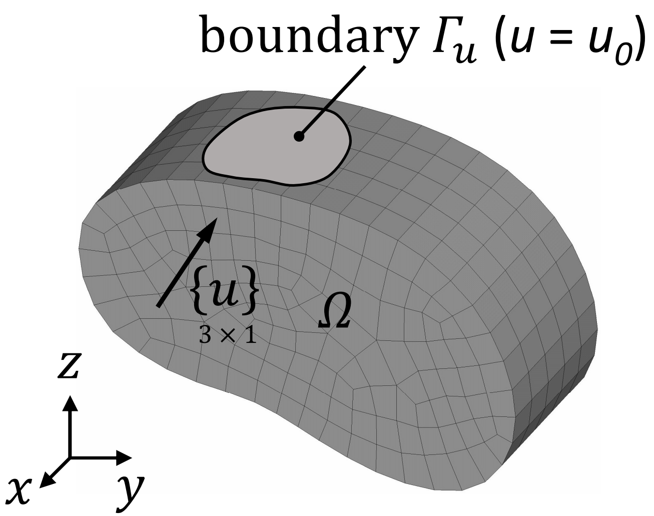
↑

set of linear algebraic equations

$$\det\left(\underset{NDOF \times NDOF}{[K]}\right) = 0$$

Set of FE equations with boundary conditions

The displacement field $\{u\}$ that represents solution of the problem fulfils displacement boundary conditons on Γ_u and minimizes the total potential energy V.



$NDOF$ – no. of degrees of freedom

NOF – no. of known degrees of freedom on Γ_u

N – number of unknown degrees of freedom:

$$N = NDOF - NOF$$

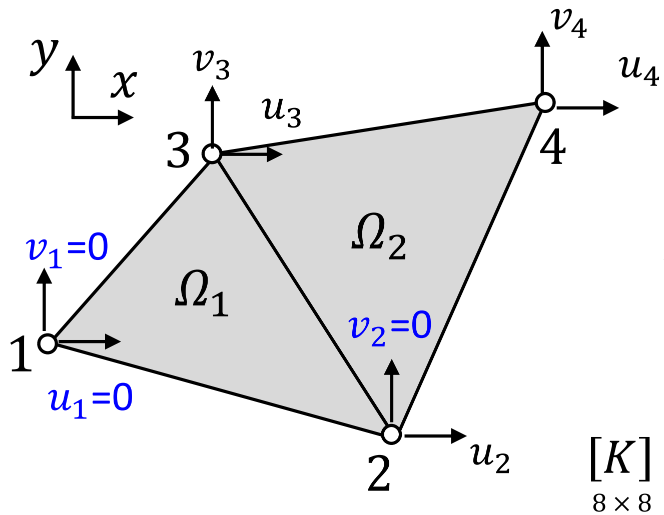
$$\underset{NDOF\times NDOF}{[K]} \rightarrow \underset{N\times N}{[K]} \;;\; \underset{NDOF\times 1}{\{q\}} \rightarrow \underset{N\times 1}{\{q\}} \;;\; \underset{NDOF\times 1}{\{F\}} \rightarrow \underset{N\times 1}{\{F\}}$$

$$\underset{N\times N}{[K]} \cdot \underset{N\times 1}{\{q\}} = \underset{N\times 1}{\{F\}}$$

$$\det\left(\underset{N\times N}{[K]}\right) \neq 0$$

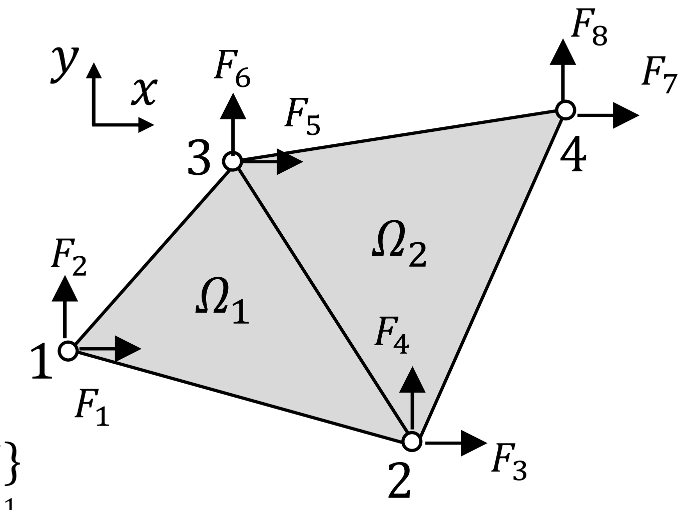
↑ *linear set of algebraic equations with boundary conditions*

Example. Boundary conditions for 2D problem. FE model with two 3-node triangles



$NDOF$ = 8

NOF = 3

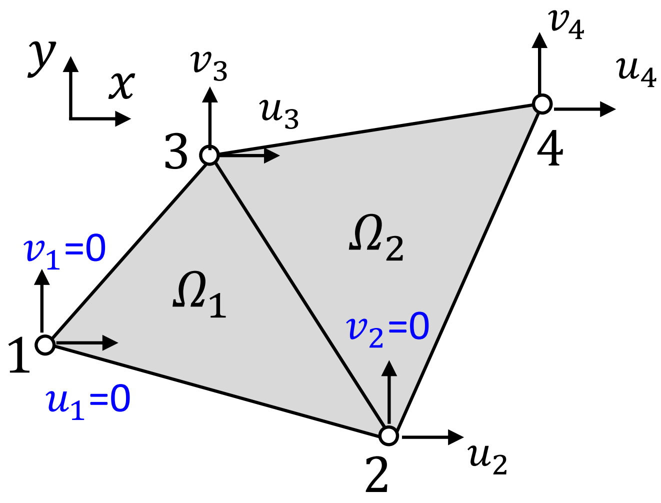


$$\underset{8\times 8}{[K]} \cdot \underset{8\times 1}{\{q\}} = \underset{8\times 1}{\{F\}}$$

	1	2	3	4	5	6	7	8
1	a_1	b_1	c_1	d_1	e_1	f_1	0	0
2	b_1	g_1	h_1	i_1	j_1	k_1	0	0
3	c_1	h_1	l_1+a_2	m_1+b_2	n_1+c_2	o_1+d_2	e_2	f_2
4	d_1	i_1	m_1+b_2	p_1+g_2	r_1+h_2	s_1+i_2	j_2	k_2
5	e_1	j_1	n_1+c_2	r_1+h_2	t_1+l_2	$\bar{u}_1+m_2$	n_2	o_2
6	f_1	k_1	o_1+d_2	t_1+l_2	$\bar{u}_1+m_2$	$\bar{w}_1+p_2$	r_2	s_2
7	0	0	e_2	j_2	n_2	r_2	t_2	$\bar{u}_2$
8	0	0	f_2	k_2	o_2	s_2	$\bar{u}_2$	$\bar{w}_2$

$$\begin{Bmatrix} u_1 = 0 \\ v_1 = 0 \\ u_2 \\ v_2 = 0 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \\ F_5 \\ F_6 \\ F_7 \\ F_8 \end{Bmatrix}$$

Example. Boundary conditions for 2D problem. FE model with two 3-node triangles

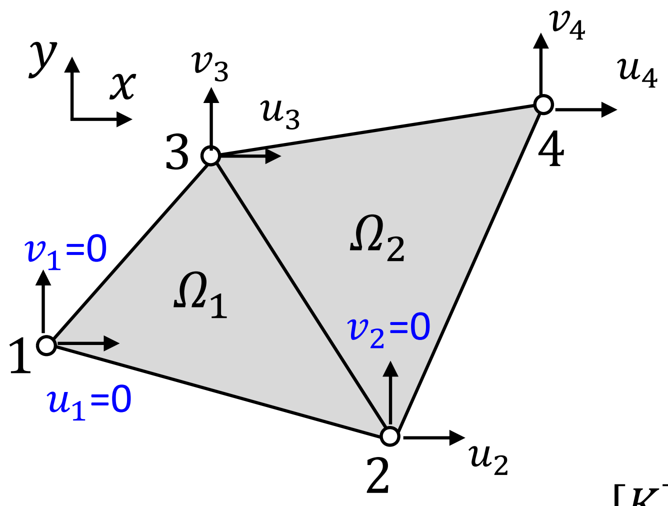


$$\underset{8\times 8}{[K]} \cdot \underset{8\times 1}{\{q\}} = \underset{8\times 1}{\{F\}}$$

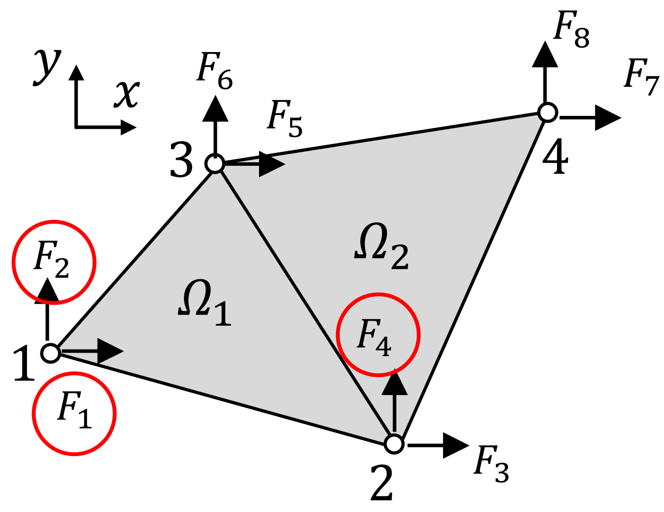
	1	2	3	4	5	6	7	8
1	a_1	b_1	c_1	d_1	e_1	f_1	0	0
2	b_1	g_1	h_1	i_1	j_1	k_1	0	0
3	c_1	h_1	l_1+a_2	m_1+b_2	n_1+c_2	o_1+d_2	e_2	f_2
4	d_1	i_1	m_1+b_2	p_1+g_2	r_1+h_2	s_1+i_2	j_2	k_2
5	e_1	j_1	n_1+c_2	r_1+h_2	t_1+l_2	$\bar{u}_1+m_2$	n_2	o_2
6	f_1	k_1	o_1+d_2	t_1+l_2	$\bar{u}_1+m_2$	$\bar{w}_1+p_2$	r_2	s_2
7	0	0	e_2	j_2	n_2	r_2	t_2	$\bar{u}_2$
8	0	0	f_2	k_2	o_2	s_2	$\bar{u}_2$	$\bar{w}_2$

$$\begin{Bmatrix} u_1 = 0 \\ v_1 = 0 \\ u_2 \\ v_2 = 0 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \\ F_5 \\ F_6 \\ F_7 \\ F_8 \end{Bmatrix}$$

Example. Boundary conditions for 2D problem. FE model with two 3-node triangles



$N = 8 - 3 = 5$



$$\underset{5\times 5}{[K]} \cdot \underset{5\times 1}{\{q\}} = \underset{5\times 1}{\{F\}}$$

$$\begin{bmatrix} l_1+a_2 & n_1+c_2 & o_1+d_2 & e_2 & f_2 \\ n_1+c_2 & t_1+l_2 & u_1+m_2 & n_2 & o_2 \\ o_1+d_2 & u_1+m_2 & w_1+p_2 & r_2 & s_2 \\ e_2 & n_2 & r_2 & t_2 & \bar{u}_2 \\ f_2 & o_2 & s_2 & \bar{u}_2 & \bar{w}_2 \end{bmatrix} \begin{Bmatrix} u_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{Bmatrix} = \begin{Bmatrix} F_3 \\ F_5 \\ F_6 \\ F_7 \\ F_8 \end{Bmatrix}$$

linear set of algebraic equations with boundary conditions

Solution of a set of FE equations with boundary conditions

$$\underset{N\times N}{[K]}\cdot\underset{N\times 1}{\{q\}} = \underset{N\times 1}{\{F\}} \quad\rightarrow\quad \det\left(\underset{N\times N}{[K]}\right)\neq 0 \quad\rightarrow\quad \underset{N\times 1}{\{q\}} = \underset{N\times N}{[K]^{-1}}\,\underset{N\times 1}{\{F\}}$$

$DOF\ solution$: $\underset{NDOF\times 1}{\{q\}}$

$Element\ solution\ (ES)$: $\underset{6\times 1}{\{\varepsilon\}} = \underset{6\times n_e}{[B]}\underset{n_e\times 1}{\{q\}_e}$; $\underset{6\times 1}{\{\sigma\}} = \underset{6\times 6}{[D]}\,\underset{6\times 1}{\{\varepsilon\}} = \underset{6\times 6}{[D]}\,\underset{6\times n_e}{[B]}\,\underset{n_e\times 1}{\{q\}_e}$

↑ strain in a finite element ↑ stress in a finite element

$Nodal\ solution\ (NS)$:

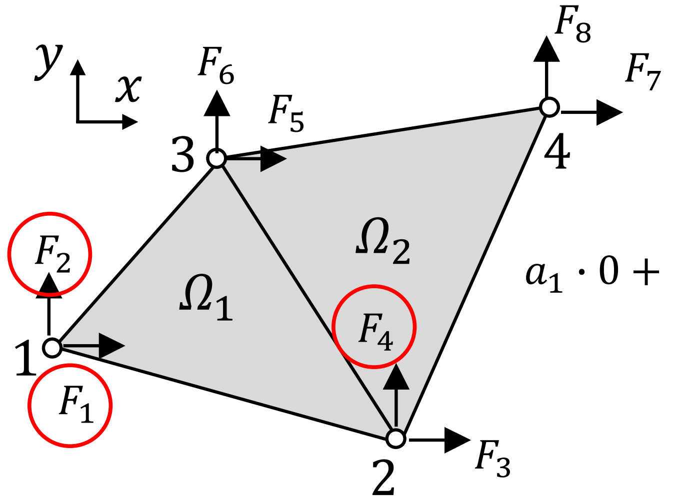
$$(NS)_i = \frac{\sum_{e=1}^{k}(ES)_{ei}}{k}$$

$(NS)_i$ − avaraged nodal solution at node (i)

$(ES)_{ei}$ − element solution in element (e) and at node (i)

k − no. of elements adjacent to node (i)

Example. Reactions calculation for 2D problem. FE model with two 3-node triangles



$$\underset{8\times 8}{[K]} \cdot \underset{8\times 1}{\{q\}} = \underset{8\times 1}{\{F\}}$$

$\{q\}$ ↑ known

$\square \cdot \square = F_1$

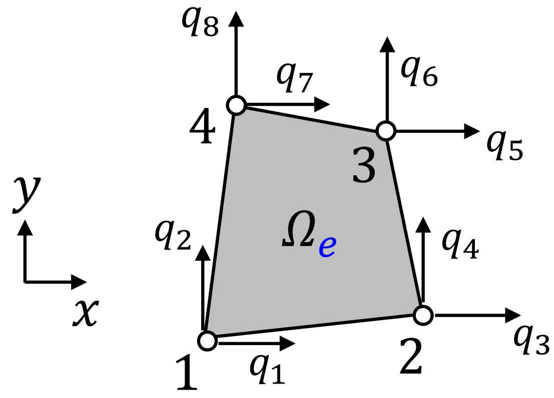
$$a_1 \cdot 0 + b_1 \cdot 0 + c_1 \cdot u_2 + d_1 \cdot 0 + e_1 \cdot u_3 + f_1 \cdot v_3 + 0 \cdot u_4 + 0 \cdot v_4 = F_1$$

$\square \cdot \square = F_2$; $\square \cdot \square = F_4$

	1	2	3	4	5	6	7	8
1	a_1	b_1	c_1	d_1	e_1	f_1	0	0
2	b_1	g_1	h_1	i_1	j_1	k_1	0	0
3	c_1	h_1	l_1+a_2	m_1+b_2	n_1+c_2	o_1+d_2	e_2	f_2
4	d_1	i_1	m_1+b_2	p_1+g_2	r_1+h_2	s_1+i_2	j_2	k_2
5	e_1	j_1	n_1+c_2	r_1+h_2	t_1+l_2	$\bar{u}_1+m_2$	n_2	o_2
6	f_1	k_1	o_1+d_2	t_1+l_2	$\bar{u}_1+m_2$	$\bar{w}_1+p_2$	r_2	s_2
7	0	0	e_2	j_2	n_2	r_2	t_2	$\bar{u}_2$
8	0	0	f_2	k_2	o_2	s_2	$\bar{u}_2$	$\bar{w}_2$

$$\begin{Bmatrix} u_1 = 0 \\ v_1 = 0 \\ u_2 \\ v_2 = 0 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \\ F_5 \\ F_6 \\ F_7 \\ F_8 \end{Bmatrix}$$

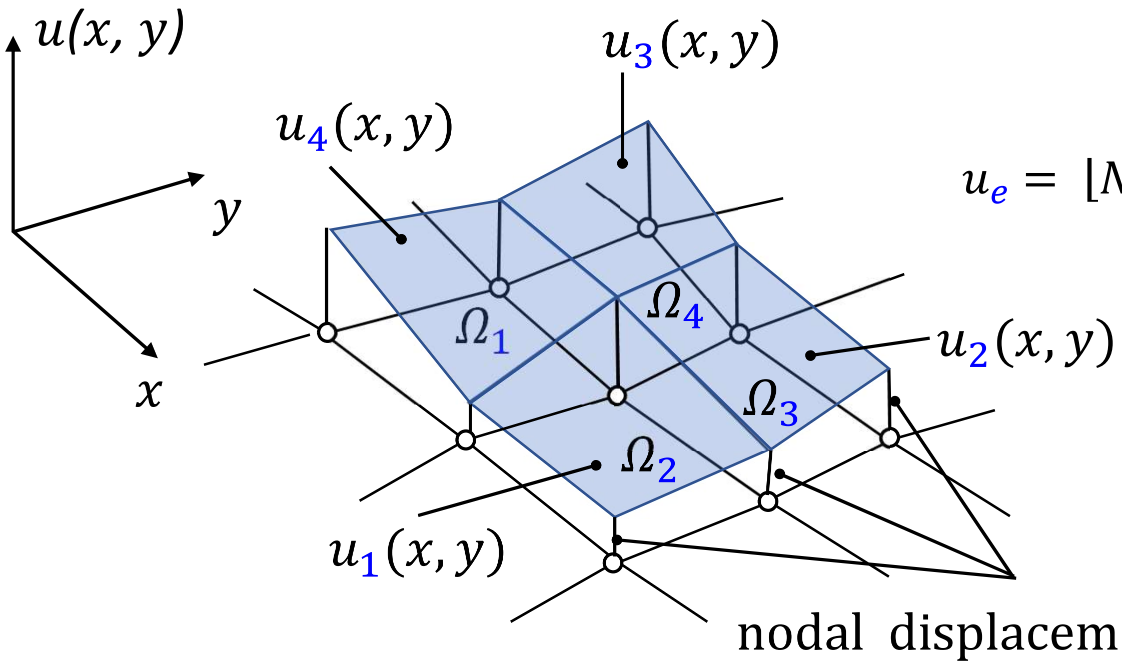
Example. DOF solution $u(x,y)$ for 2D problem. FE model with 4-node quadrilateral elements



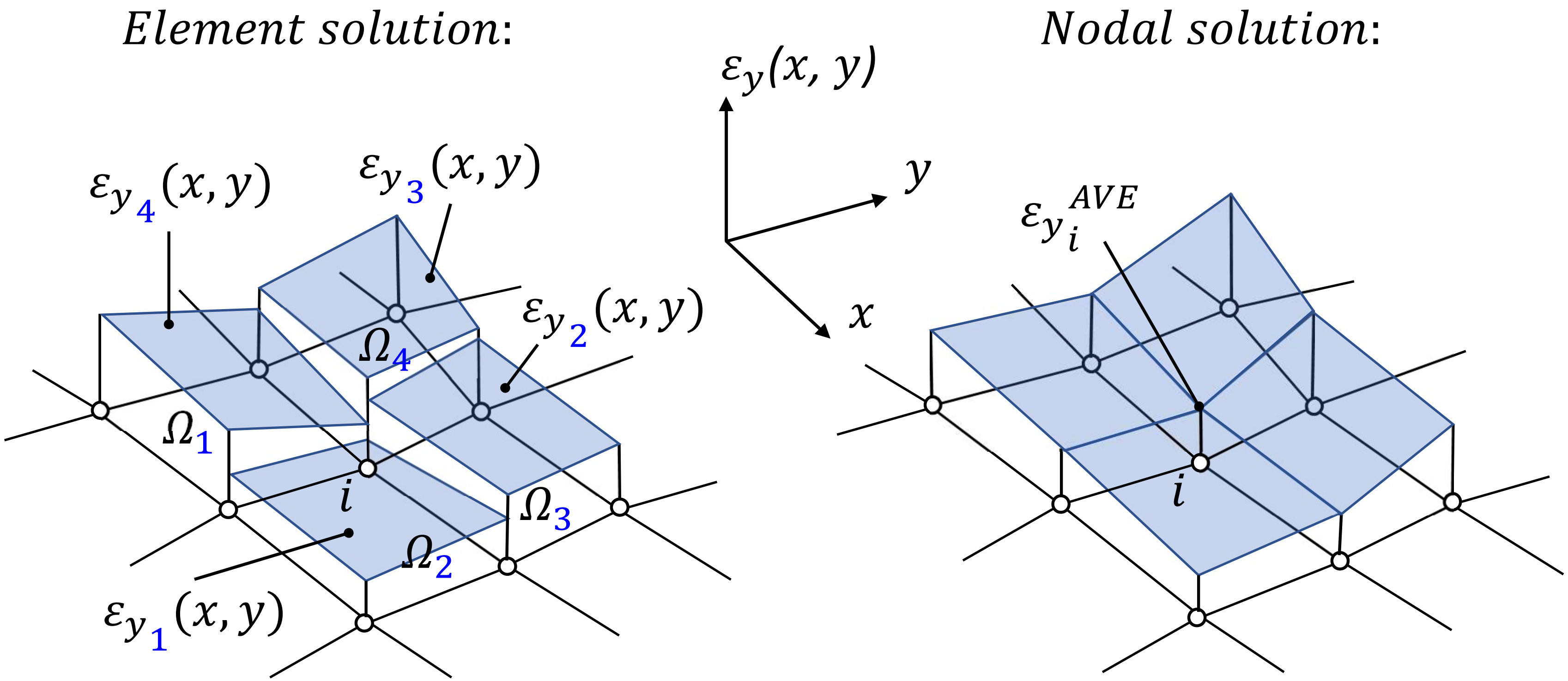
$$\underset{2\times 1}{\{u\}} = \underset{2\times 8}{[N]}\underset{8\times 1}{\{q\}_e}$$

$u_e(x,y)$ − displacement in x direction

$$u_e = [N_1\ 0\ \ N_2\ 0\ \ N_3\ 0\ \ N_4\ 0]\begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \\ q_6 \\ q_7 \\ q_8 \end{Bmatrix}_e$$



Example. Strain component $\varepsilon_y(x,y)$ for 2D problem. FE model with 4-node quadrilateral elements



$k = 4$

$$\varepsilon_{y_i}^{AVE} = \frac{\varepsilon_{y_1}(x_i, y_i) + \varepsilon_{y_2}(x_i, y_i) + \varepsilon_{y_3}(x_i, y_i) + \varepsilon_{y_4}(x_i, y_i)}{4}$$